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John Stansfeld

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TREATISE

ON

ALGEBRAICAL EQUATIONS.

A TREATISE

ON THE

THEORY

OF

ALGEBRAICAL EQUATIONS.

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THEORY OF EQUATIONS.

SECTION I.

ON THE GENERAL PROPERTIES OF EQUATIONS.

1. EVERY equation under a rational form involving the powers of only one unknown quantity x, may, by dividing its two members by the coefficient of the highest power of x, and transposing the terms, be reduced to the form

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0;$$

where x^n, the highest power of x, is positive and its coefficient unity; and $p_1, p_2, \ldots p_n$, the coefficients of the other powers of x, are known quantities which may be positive, or negative, or zero. The equation is said to be of the number of dimensions, or of the degree, which is expressed by the index of the highest power of x which it involves; and to be complete, when it contains all the other inferior powers of x, and a constant term; otherwise, to be incomplete.

2. Every quantity or expression, real or imaginary, which, when substituted for x in the expression $x^n + p_1x^{n-1} + \ldots + p_n$, makes the whole vanish, is called a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0.$$

To solve an equation is to find all its roots. To do this generally, would be to find the expressions for all the roots of an equation of any assigned degree in terms of its coefficients, the coefficients being general symbols. This has hitherto been effected only for equations which do not exceed the 4th degree; and even for cubic equations, the functions of the coefficients which express the roots are insufficient to give the numerical values of the roots when they are all real; hence we are led to suppose that if we could obtain general formulæ for the roots of equations of the 5th and superior degrees, we should be unable to obtain from them the numerical values of the roots by the simple substitution of those of the coefficients. It has therefore become necessary to invent methods for obtaining, either exactly or approximately, the roots of numerical equations, and which, although only applicable to such equations, depend for their demonstration upon certain general properties of equations, which it is the object of the following articles to exhibit.

3. If the signs of the terms of any equation be all positive, it cannot have a positive root; and if the signs of a complete equation be alternately positive and negative, it cannot have a negative root.

For in the former case, every positive quantity substituted for x will give a positive result, instead of making the whole vanish, and therefore cannot be a root; and in the latter case, every negative quantity substituted for x will give a positive or negative result, according as the degree of the equation is even or odd, instead of making the whole vanish, and therefore cannot be a root.

4. If a be a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0,$$

the first member is divisible by $x - a$ without a remainder, and conversely.

Suppose the expression $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n$, which we shall hereafter denote by $f(x)$, to be divided by $x - a$; now since $x - a$ is only of one dimension with respect to x, the division may be carried on till we obtain a remainder R independent of x; let Q be the quotient in which only positive powers of x will enter,

$$\therefore f(x) = \mathrm{Q}.\,(x - a) + \mathrm{R} \ldots (1).$$

In this identical equation write a for x, then the first member becomes zero, because a is a root of $f(x) = 0$; also the term $\mathrm{Q}\,.\,(x - a)$ vanishes, since one of its factors vanishes and the other cannot become infinite; therefore $\mathrm{R} = 0$, and since R does not contain x, it is not altered by substituting a for x, and therefore zero is the value of R in equation (1), whatever be the value of x; that is, $f(x)$ is exactly divisible by $x - a$.

Conversely, if the expression $x^n + p_1x^{n-1} + \ldots + p_n$ be divisible by $x - a$ without a remainder, a is a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0.$$

For $f(x) = \mathrm{Q}\,.\,(x - a)$, where Q is a polynomial containing only positive powers of x; if therefore $x = a$, $f(a) = 0$, or a is a root of the equation $f(x) = 0$.

5. Hence, since a is visibly a root of $x^n - a^n = 0$, $x^n - a^n$ is divisible by $x - a$, whether n be odd or even, and

$$\text{the quotient} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \ldots + a^{n-1};$$

also when n is even, $x^n - a^n$ is divisible by $x + a$, since in that case $-a$ is a root of $x^n - a^n = 0$.

When n is odd, $-a$ is a root of $x^n + a^n = 0$; therefore in this case $x^n + a^n$ is divisible by $x + a$, but not by $x - a$; if n is even, $x^n + a^n$ is divisible by neither $x+a$ nor $x-a$.

6. To find the quotient and remainder, when the expression $x^n + p_1x^{n-1} + \ldots + p_n$ is divided by $x - a$, a being any quantity.

Let the division be carried on till the remainder is independent of x, and let Q be the quotient and R the remainder;

$\therefore\ x^n+p_1x^{n-1}+p_2x^{n-2}+\ \ldots\ + p_n = \mathrm{Q}\,(x-a) + \mathrm{R}\ldots(1),$
in which identical equation, since R does not contain x, and Q contains only positive powers, if we write a for x, we have

$$a^n + p_1a^{n-1} + p_2a^{n-2} + \ldots + p_n = \mathrm{R};$$

that is, the remainder, after dividing $f(x)$ by $x - a$, is equal to $f(a)$, the value assumed by $f(x)$ when in it a is written for x.

Next, substituting this value of R in equation (1) and transposing, we have

$$x^n - a^n + p_1(x^{n-1} - a^{n-1}) + p_2(x^{n-2} - a^{n-2}) + \ldots$$
$$+ p_{n-1}(x - a) = \mathrm{Q}\,(x - a);$$

but the quantities $x^n - a^n$, $x^{n-1} - a^{n-1}$, ... are all divisible by $x - a$; therefore, effecting the division, we have

$$x^{n-1}+ax^{n-2}+a^2x^{n-3}+\ \ldots\ +a^{n-1}$$
$$+p_1(x^{n-2}+ax^{n-3}+a^2x^{n-4}+\ \ldots\ +a^{n-2})+\ \ldots\ +p_{n-1}=\mathrm{Q},$$

or, arranging the result according to powers of x,

$$\mathrm{Q} = x^{n-1}+(a+p_1)x^{n-2}$$
$$+(a^2+ap_1+p_2)\,x^{n-3}+(a^3+a^2p_1+ap_2+p_3)\,x^{n-4}+\ldots$$
$$+(a^{n-1}+p_1a^{n-2}+p_2a^{n-3}+\ldots+p_{n-1});$$

that is, in the quotient of $f(x)$ divided by $x - a$, the coefficient of the first term x^{n-1} is unity, and the other coefficients

are formed, one from the other, by multiplying the preceding coefficient by a, and adding the coefficient of that term in $f(x)$ which involves the same power of x as the preceding coefficient does.

7. If two quantities a and b, when substituted for x in the expression $f(x)$, give results affected with different signs, one root at least of the equation $f(x) = 0$ lies between them.

Suppose $a < b$, and suppose a to give a positive result, and b a negative result when substituted for x in the expression $f(x)$. Let P be the sum of the positive, N the sum of the negative terms in $f(x)$; then when $x = a$, $\mathrm{P} - \mathrm{N}$ is positive or $\mathrm{P} > \mathrm{N}$, and when $x = b$, $\mathrm{P} - \mathrm{N}$ is negative or $\mathrm{P} < \mathrm{N}$; let x change by insensible degrees from a to b, then P and N both increase, but P increases slower than N since when $x = b$, $\mathrm{P} < \mathrm{N}$; consequently, for some intermediate value of x between a and b, $\mathrm{P} = \mathrm{N}$ or $\mathrm{P} - \mathrm{N} = 0$ or $f(x)$ becomes equal to zero; this value therefore is a root of the equation. If the smaller quantity a gave a negative result, the proof would be precisely similar.

Also since the first member of the equation may pass several times from positive to negative, or from negative to positive, by the substitution of gradually ascending values between a and b, it follows that several roots of $f(x) = 0$ may be comprised between a and b, and we are certain that one is.

8. Hence, if there exist no real quantity which substituted for x will make $f(x)$ vanish, $f(x)$ must be positive for every value of x; for if it became negative for any value, since by substituting ∞ for x in

$$f(x) = x^n\left(1 + \frac{p_1}{x} + \frac{p_2}{x^2} + \ldots . + \frac{p_n}{x^n}\right),$$

we necessarily obtain a positive result, the equation $f(x) = 0$ would have a real root, which is contrary to the supposition.

9. It is always possible to assign such a finite value to x, as will make the first term of $x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n$ greater than the sum of all the other terms.

Let p be the greatest coefficient without regard to signs; then if

$$x^n > p(x^{n-1} + x^{n-2} + \ldots . + x + 1) > p\frac{x^n - 1}{x - 1},$$

we shall of course have

$$x^n > p_1x^{n-1} + p_2x^{n-2} + \ldots . + p_{n-1}x + p_n,$$

because in the latter inequality some of the terms may be negative, and no positive term is greater than the corresponding term in the former case. Now the inequality $x^n > p\frac{x^n - 1}{x - 1}$ is satisfied if $x^n =$ or $> x^n\frac{p}{x - 1}$, or if $x =$ or $> p + 1$; therefore $p + 1$, and every greater number, is a value of x which makes the first term of $x^n + p_1x^{n-1} + \ldots . + p_n$ greater than the sum of all the other terms.

Again, let $x = -y$, then, according as n is even or odd,

$$x^n + p_1x^{n-1} + \ldots . + p_{n-1}x + p_n \text{ becomes}$$
$$y^n - p_1y^{n-1} + \ldots . - p_{n-1}y + p_n,$$
$$\text{or } -(y^n - p_1y^{n-1} + \ldots . + p_{n-1}y - p_n).$$

Now by what has already been proved, the value $p + 1$, and every greater value for y, makes the former expression positive, and the latter negative; therefore the value $-(p + 1)$, or any greater negative value for x, makes $x^n + p_1x^{n-1} + \ldots + p^n$ positive or negative, according as n is even or odd.

10. Every equation of an odd degree has at least one real root of a sign contrary to that of its last term; and every equation of an even degree with its last term negative has at least two real roots of different signs.

First, let the equation be of an odd degree with its last term negative; then $x = 0$ gives a negative result, and $x = p + 1$ gives a positive result (p being the greatest coefficient without

regard to signs); therefore the equation has at least one real positive root between 0 and $p+1$. If the last term be positive, then $x=0$ gives a positive result, and $x=-(p+1)$ gives a negative result; therefore the equation has at least one real negative root between 0 and $-(p+1)$.

Secondly, let the equation be of an even degree with its last term negative; then $x=0$ gives a negative result, and each of the values $x=p+1$, $x=-(p+1)$ gives a positive result; therefore the equation has at least two real roots, one positive between 0 and $p+1$, and the other negative between 0 and $-(p+1)$.

11. Hence we are certain of the existence of a real root in every equation unless it be of an even degree with its last term positive, in which case it may have no real root; but then there may, and, as will hereafter be shewn, must exist an impossible expression of the form $a+\beta\sqrt{-1}$ (a and β being possible quantities) which substituted for x in $f(x)$ will make the whole vanish. We shall therefore for the present assume that every equation admits a root of the form $a+\beta\sqrt{-1}$, a and β being real finite quantities, or either of them being zero; that is, we shall assume, not only that every equation has a root expressible by algebraical symbols, but that $a+\beta\sqrt{-1}$ is the form which the root necessarily takes.

12. Every equation has as many roots as it has dimensions, and no more.

Since every equation has necessarily a root real or imaginary, let a_1 be a root of $f(x)=0$; then $f(x)$ is divisible by $x-a_1$, let $f_1(x)$ be the quotient,

$$\therefore f(x)=(x-a_1)f_1(x),$$

$f_1(x)$ denoting a polynomial of $n-1$ dimensions exactly similar to $f(x)$, and having therefore the same properties. Hence

$f_1(x) = 0$ must have a root real or imaginary; let this be a_2, and let $f_2(x)$, a polynomial of $n - 2$ dimensions, be the quotient of $f_1(x)$ by $x - a_2$;

$$\therefore f_1(x) = (x - a_2) f_2(x),$$

$$\text{and } f(x) = (x - a_1)(x - a_2) f_2(x)$$

Similarly, $f_2(x) = (x - a_3) f_3(x)$, and proceeding in this manner we shall at last come to a quotient of only one dimension in x, so that

$$f_{n-2}(x) = (x - a_{n-1}) f_{n-1}(x) = (x - a_{n-1})(x - a_n),$$

therefore, by successive substitutions, we have

$$f(x) = (x - a_1)(x - a_2)(x - a_3) \ldots (x - a_{n-1})(x - a_n).$$

Now in order that the product of n simple factors may vanish, it is sufficient that any one of the factors should vanish; therefore we shall satisfy the equation $f(x) = 0$, by giving to x any one of the n values $a_1, a_2, a_3, \ldots . a_n$. Neither can we satisfy it by any other values, for if possible let e be a root of $f(x) = 0$, e being different from the quantities $a_1, a_2, \ldots . a_n$; then $f(e)$ or $(e - a_1)(e - a_2) \ldots . (e - a_n)$ must be equal to zero, but this is impossible since not one of the factors is so; therefore e is not a root. Therefore every equation of the n^{th} degree has n roots, and no more; and every polynomial of the n^{th} degree is resolvable into one determinate system of n simple factors.

13. In the above proposition the divisors are not necessarily different from one another; any number or all of them may be equal: and it is in this sense that an equation is said to have as many roots as it has dimensions, namely, that its first member is resolvable into as many simple factors, equal or unequal, as it has dimensions, each of which equated to zero will furnish a root; so that as many times as any factor $x - a$ occurs in its first member, so many roots equal to a will the equation have.

14. Hence, if we know a root a of the equation $f(x) = 0$, we may divide the first member by $x - a$, and the quotient put equal to zero will be an equation, one dimension lower, containing the remaining roots; or we may form the reduced equation immediately, without the trouble of division, by the rule of Art. 6. Similarly, if we know two roots a and b of $f(x) = 0$, by dividing its first member by $(x - a)(x - b)$, we shall obtain an equation two dimensions lower containing the remaining roots. And, in general, if we know $n - r$ roots of $f(x) = 0$, by dividing $f(x)$ by the product of the simple factors corresponding to these roots, we may form the reduced equation of r dimensions, $\phi(x) = 0$, containing the remaining roots; and if $f(x) = 0$ has only $n - r$ real roots, then all the roots of $\phi(x) = 0$ are imaginary, and in this case $\phi(x)$ is a polynomial of an even number of dimensions with its last term positive, and is incapable of being made negative by any value of x. Hence also, if all the real roots $a_1, a_2, \ldots . a_{n-r}$, of an equation of n dimensions have been obtained, the equation will be

$$(x - a_1)(x - a_2) \ldots . (x - a_{n-r}) \cdot \phi(x) = 0,$$

where $\phi(x)$ is such as has been described.

15. Impossible roots enter equations by pairs, each pair corresponding to a real quadratic factor of the polynomial forming the first member.

Let $\alpha + \beta\sqrt{-1}$ represent one of the imaginary roots, and let it be substituted for x in the first member of the equation $f(x) = 0$. The result will consist of two parts, possible quantities which involve the powers of α and the even powers of $\beta\sqrt{-1}$, and impossible quantities which involve the odd powers of $\beta\sqrt{-1}$; let P be the sum of the possible quantities, and $Q\beta\sqrt{-1}$ that of the impossible quantities, therefore the

whole result is $P + Q\beta\sqrt{-1}$, where P and Q contain only even powers of β.

Now since $a + \beta\sqrt{-1}$ is a root,

$$P + Q\beta\sqrt{-1} = 0,$$

and as no part of P can be destroyed by $Q\beta\sqrt{-1}$, this resolves itself into $P = 0$, $Q = 0$. Now for x substitute $a - \beta\sqrt{-1}$, or change the sign of β in the former result; then since P and Q contain only even powers of β, the result is $P - Q\beta\sqrt{-1}$, which, since $P = 0$, $Q = 0$, is equal to zero; therefore $a - \beta\sqrt{-1}$ is a root of $f(x) = 0$. Therefore the proposed equation admits a pair of conjugate roots $a + \beta\sqrt{-1}$ and $a - \beta\sqrt{-1}$; and its first member admits the two factors

$$x - a - \beta\sqrt{-1},\ x - a + \beta\sqrt{-1},$$

and will therefore be divisible by their product which

$$= (x - a)^2 + \beta^2 \text{ or } x^2 - 2ax + a^2 + \beta^2.$$

In the same manner it might be shewn that when the coefficients are rational, surd roots of the form $a \pm \sqrt{b}$ enter equations by pairs.

16. Hence the whole number of impossible roots in any equation will always be even, and every equation of an even degree may be resolved into real factors of the second degree; for every pair of impossible roots will produce a real quadratic factor; and the possible roots, since there is an even number of them, may also be divided into pairs each of which will produce a real factor of the second degree.

17. Since $f(x)$, a polynomial of the n^{th} degree, always admits n divisors real or imaginary of the first degree, it will admit $\frac{n(n-1)}{1.2}$ different divisors real or imaginary of the

second degree, $\frac{n(n-1)(n-2)}{1.2.3}$ of the third degree, ... ; and in general it will admit $\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}$ of the r^{th} degree, as each of these will be a combination of r out of the n simple factors. Also the total number of divisors of all degrees will be $2^n - 1$.

18. To actually decompose the first member of a given equation $f(x) = 0$ into its real, simple or quadratic factors, is the great problem to the solution of which all enquiries in this subject are directed; but the inverse problem, to form the equation when the roots are given, offers no difficulty; for knowing the component factors of the polynomial forming its first member, we can determine that polynomial by the common process of multiplication. Thus to form the equation whose roots are $a, -b, c, c, a \pm \beta\sqrt{-1}$, we must multiply together the factors $x-a$, $x+b$, $(x-c)^2$, $x^2-2ax+a^2+\beta^2$.

19. To find the relations between the coefficients and roots of an equation.

We must first ascertain the law of formation of the products of any number of binomial factors $x+a$, $x+b$, $x+c$, . . . which have all the same first term x, but different second terms $a, b, c, \ldots\ldots$ By actual multiplication we have

$$(x+a)(x+b) = x^2+(a+b)x+ab$$

$$(x+a)(x+b)(x+c) = x^3+(a+b+c)x^2 + (ab+ac+bc)x+abc$$

$$(x+a)(x+b)(x+c)(x+d) = x^4+(a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 + (abc+abd+acd+bcd)x+abcd.$$

In these products we observe that the index of x diminishes

by unity in each term, from the first term where it is the same as the number of factors to the last where it is zero; also the coefficient of the first term is unity, that of the second is the sum of the second terms of the binomial factors, that of the third term is the sum of the products of every two, that of the fourth term is the sum of the products of every three, and the last term is the product of all the second terms of the binomial factors. To prove these laws of the indices and coefficients generally true, we must shew that if they be true for $n-1$ factors, they will be true for n factors. Let therefore the product of $n-1$ factors

$$(x+a)(x+b)(x+c)\dots(x+k) = x^{n-1}+S_1x^{n-2}+S_2x^{n-3}+\dots$$
$$+S_{r-1}x^{n-r}+\dots\dots+S_{n-1},$$

where S_1, S_2 .... denote the sum, the sum of the products of every two, ... of the $n-1$ quantities $a, b, c \dots k$. Now introduce another factor $x+l$, and we find for result

$$x^n+(S_1+l)\,x^{n-1}+(S_2+lS_1)\,x^{n-2}+\dots\dots$$
$$+(S_r+lS_{r-1})\,x^{n-r}+\dots\dots+lS_{n-1}.$$

With respect to the indices the law is unchanged; with respect to the coefficients, that of the first term is still unity, that of the

$2\text{nd} = S_1 + l =$ sum of the n quantities $a, b, c, \dots l$,

that of the

$3\text{rd} = S_2 + lS_1 =$ sum of the products of every two,

that of the

$(r+1)^{th} = S_r + lS_{r-1} =$ sum of the products of every r,

and the last term

$= lS_{n-1} =$ the product of the n quantities.

If therefore the law of formation of the product be true for $n-1$ factors, it is true for n; but it is verified for 2, 3 .... factors, therefore it is generally true.

Now let $a, b, c, \ldots . l$ be the n roots of the equation $f(x) = 0$;

$$\therefore x^n + p_1x^{n-1} + \ldots + p_rx^{n-r} + \ldots + p_n = (x-a)(x-b)(x-c)\ldots$$
$$(x-l) = x^n + S_1x^{n-1} + S_2x^{n-2} + \ldots . + S_rx^{n-r} + \ldots + S_n,$$

where $S_1, S_2, \ldots .$ denote the sums of the various combinations (taken singly, two and two, &c.) of $-a, -b, \ldots .$ that is, of the roots with their signs changed; therefore, equating coefficients,

$$p_1 = S_1, \; p_2 = S_2 \ldots . \; p_r = S_r, \; p_n = S_n,$$

or coefficient of 2nd term with its proper sign = sum of the roots with their signs changed,

coefficient of 3rd term with its proper sign = sum of the products of every two roots with their signs changed,

coefficient of the $(r+1)^{th}$ term with its proper sign = sum of the products of every r roots with their signs changed,

and the last term with its proper sign = the product of all the roots with their signs changed.

Or, if we choose, which is more convenient, to employ in the enunciation both the roots and the coefficients with their proper signs, we have

$-p_1 =$ sum of the roots, $p_2 =$ sum of the products of every two, $-p_3 =$ sum of the products of every three, and generally, $(-1)^r p_r =$ sum of the products of every r roots.

20. These relations, which furnish n equations between the roots and the coefficients, do not afford any immediate means of finding the roots; and if we wished to employ them to find one of the roots by the elimination of the $n-1$ others, we should always arrive at an equation similar to the proposed.

Let, for example, the equation be of the third degree,

$$x^3 + p_1x^2 + p_2x + p_3 = 0, \quad \text{roots } a, b, c;$$

$$\therefore p_1 = \quad (a + b + c)$$
$$p_2 = ab + ac + bc$$
$$p_3 = -abc,$$

to eliminate b and c between these equations, multiply the first by a^2, and the second by a, and add them to the third, and we find

$$a^3 + p_1 a^2 + p_2 a + p_3 = 0.$$

21. But although not leading to the determination of the roots, the above relations will enable us to discover many of their properties, and are to be regarded as constituting one of the fundamental propositions of the theory. At present we shall employ them to find the values of some of the more common symmetrical functions of the roots; that is, of functions in which each root is alike involved, so that their values are unaltered when any two of the roots are interchanged.

(1.) To find the sum of the squares of the roots of $f(x) = 0$.

$$-p_1 = a + b + c + \ldots\ldots + l;$$
$$\therefore p_1^2 = a^2 + b^2 + c^2 + \ldots\ldots + l^2 + 2(ab + ac + bc + \ldots\ldots)$$
$$= \text{sum of squares} + 2p_2;$$
$$\therefore \text{sum of squares} = p_1^2 - 2p_2.$$

(2.) To find the sum of the reciprocals of the roots.

$$(-1)^{n-1} p_{n-1} = bc \ldots l + ac \ldots l + ab \ldots l + \ldots\ldots$$
$$(-1)^n p_n = abc \ldots\ldots l;$$
$$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots\ldots + \frac{1}{l} = -\frac{p_{n-1}}{p_n}.$$

(3.) To find the sum of $\frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \ldots\ldots$

$$\text{This} = a\left(\frac{1}{a}+\frac{1}{b}+\ldots+\frac{1}{l}\right)-1+b\left(\frac{1}{a}+\frac{1}{b}+\ldots+\frac{1}{l}\right)-1+\ldots$$

$$=(a+b+\ldots+l)\left(\frac{1}{a}+\frac{1}{b}+\ldots+\frac{1}{l}\right)-n$$

$$=(-p_1)\left(-\frac{p_{n-1}}{p_n}\right)-n=\frac{p_1p_{n-1}}{p_n}-n.$$

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22. The following are examples of depressing an equation when one or more of its roots are known, or of forming it when all its roots are known; also of resolving certain polynomials into their factors.

(1.) To find the roots of $x^3 - 1 = 0$.

One root is $x = 1$; dividing by $x - 1$, we get the quadratic $x^2 + x + 1 = 0$ containing the other two roots, and which gives for their values

$$x = \frac{-1 \pm \sqrt{-3}}{2}.$$

(2.) To find the roots of $x^5 - 9x^3 + x^2 - 9 = 0$.

The first member $= x^3(x^2-9) + x^2-9 = (x^3+1)(x^2-9)$ $= (x+1)(x^2-x+1)(x+3)(x-3)$;

$\therefore$ the five values of x are $-1, -3, 3, \dfrac{1 \pm \sqrt{-3}}{2}$.

(3.) A root of $x^9+x^8-9x^7+3x^6-8x^5+8x^4-3x^3+9x^2-x-1=0$ is unity, to form the equation containing the remaining roots, it is

$$x^8+(1+1)x^7+(2-9)x^6+(-7+3)x^5+(-4-8)x^4$$
$$+(-12+8)x^3+(-4-3)x^2+(-7+9)x+(2-1)=0;$$

or $x^8+2x^7-7x^6-4x^5-12x^4-4x^3-7x^2+2x+1=0$.

(4.) To form the equation whose roots are

$$4,\ -1,\ \tfrac{1}{2}(-3\pm\sqrt{-31}).$$

It is $(x-4)(x+1)(x^2+3x+10)=0$;

$$\text{or } x^4-3x^2-42x-40=0.$$

(5.) The equation of eight dimensions (in which the coefficients are dependent upon one another by particular relations)

$$x^8+4nx^6+2x^4-4nx^2+1=0,$$

may be solved as a quadratic; and by reason of the double values of the radical quantities involved, the eight roots are expressed by one formula

$$x=\sqrt{(\sqrt{n+1}-\sqrt{n})(\sqrt{n-1}+\sqrt{n})}.$$

(6.) The preceding is an instance of what must happen whenever the general solution of any equation can be effected, as stated in Art. 2. We shall next give an example of an equation of the n^{th} degree, where it is possible to get a formula expressing the n roots and no other quantities, viz. the binomial equation

$$x^n \pm 1 = 0.$$

This is the only extensive class of equations that has been solved by a general method; and the discussion of the nature and properties of their roots is of great interest and importance in the theory of equations. It is convenient to consider the two cases $x^n-1=0$ and $x^n+1=0$ separately.

(7.) All the roots of $x^n-1=0$ are impossible, except one when n is odd, and two when n is even.

If we expel the factors $x-1$ or x^2-1 according as n is odd or even, the depressed equations are

$$x^{n-1}+x^{n-2}+\ldots+x+1=0,$$

$$x^{n-2}+x^{n-4}+\ldots+x^2+1=0,$$

of which the former cannot have a positive root, and it cannot have a negative root since the proposed cannot have a negative

root, n being odd; and the latter, since it contains only even powers of x, can neither have a positive nor a negative root; therefore the depressed equations have all their roots impossible. Since the proposed equation is the same as $x^n = 1$, the condition which it expresses is, that the arithmetical or algebraical values of x are such, that being raised to the n^{th} power they produce unity. On this account the roots of $x^n - 1 = 0$ are called the n^{th} roots of unity.

(8.) To solve the equation $x^n - 1 = 0$.

Since the equation can only have the real roots 1 and -1, we may assume $x = \cos\theta \pm \sqrt{-1}\sin\theta$, for this value will coincide with the real roots when θ is zero or a multiple of π, and in all other cases will be imaginary. Then *De Moivre's* formula gives

$$x^n = \cos n\theta \pm \sqrt{-1}\sin n\theta;$$

therefore all values of θ determined by the condition

$$\cos n\theta \pm \sqrt{-1}\sin n\theta = 1,$$

will give values of x which are roots of the proposed; therefore we must separately have $\sin n\theta = 0$, $\cos n\theta = +1$, and consequently $n\theta$ must be an even multiple of π, $= 2\lambda\pi$ suppose, where λ is any integer whatever. Hence all values of x comprised in the formula

$$x = \cos\frac{2\lambda\pi}{n} \pm \sqrt{-1}\sin\frac{2\lambda\pi}{n} \quad . \; . \; . \; . \quad (1)$$

are roots of $x^n - 1 = 0$, or are n^{th} roots of unity.

Moreover this expression has n different values and no more.

For, taking λ from zero to $\frac{1}{2}(n-1)$ or $\frac{1}{2}n$ according as n is odd or even, we find in the first case, one real value $+1$ when $\lambda = 0$, and $\frac{1}{2}(n-1)$ pairs of imaginary values corres-

ponding to values of λ from 1 to $\frac{1}{2}(n-1)$, or n values on the whole; and in the second case, we find one real value $+1$ when $\lambda = 0$, one real value -1 when $\lambda = \frac{1}{2}n$, and $\frac{1}{2}n - 1$ pairs of imaginary values corresponding to values of λ from 1 to $\frac{1}{2}n-1$, or n values on the whole.

And all these imaginary values are different from one another, because the series of angles involved in them

$$\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{6\pi}{n}, \ldots \ldots \frac{(n-1)\pi}{n} \text{ or } \frac{(n-2)\pi}{n} \quad \ldots \ (2)$$

are all different from one another and all less than π.

Also the formula (1) has no more than n values.

For if we take λ negative, the two values are not altered but only interchanged; and if we take $\lambda =$ or $> n$, the effect is to add a multiple of 2π to one of the angles (2) which alters neither the cosine nor sine; and lastly, if we consider the values of x corresponding to values of λ, m and $n-m$ equally distant from 0 and n, we shall find them the same; for taking $\lambda = n - m$,

$$x = \cos\frac{2(n-m)\pi}{n} \pm \sqrt{-1}\sin\frac{2(n-m)\pi}{n}$$

$$= \cos\frac{-2m\pi}{n} \pm \sqrt{-1}\sin\frac{-2m\pi}{n} = \cos\frac{2m\pi}{n} \mp \sqrt{-1}\sin\frac{2m\pi}{n},$$

the same as when $\lambda = m$; so that we can get no new values by taking λ greater than $\frac{1}{2}n$.

Therefore the formula can never assume any other values than the n different ones resulting from taking λ from 0 to $\frac{1}{2}(n-1)$ or $\frac{1}{2}n$, according as n is odd or even; since therefore the formula

$$x = \cos\frac{2\lambda\pi}{n} \pm \sqrt{-1}\sin\frac{2\lambda\pi}{n}$$

equally expresses all the roots of $x^n - 1 = 0$, and no other quantities, it is the complete solution of that equation.

(9.) Hence we observe that for any value of λ, except zero, or $\frac{1}{2}n$ when n is even, the two corresponding roots are conjugate, and one is the reciprocal of the other; for

$$\left(\cos\frac{2\lambda\pi}{n} + \sqrt{-1}\sin\frac{2\lambda\pi}{n}\right)\left(\cos\frac{2\lambda\pi}{n} - \sqrt{-1}\sin\frac{2\lambda\pi}{n}\right) = 1.$$

Also since

$$\cos\frac{2\lambda\pi}{n} \pm \sqrt{-1}\sin\frac{2\lambda\pi}{n} = \left(\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}\right)^{\pm\lambda}$$

we observe this remarkable relation among the imaginary roots, that they are all powers of the first imaginary root corresponding to $\lambda = 1$, viz. $\cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}$; so that if we denote this by a, the series of imaginary roots will be

$$a,\ a^2,\ a^3 \ldots\ldots a^{\frac{n-1}{2}} \text{ or } a^{\frac{n-2}{2}},$$

$$\frac{1}{a},\ \frac{1}{a^2},\ \frac{1}{a^3} \ldots\ldots \frac{1}{a^{\frac{n-1}{2}}} \text{ or } \frac{1}{a^{\frac{n-2}{2}}},$$

or since $a^n = 1$, the lower line may be replaced by

$$a^{n-1},\ a^{n-2} \ldots\ldots a^{\frac{n+1}{2}} \text{ or } a^{\frac{n+2}{2}};$$

therefore, since $a^{\frac{n}{2}} = -1$ when n is even, all the roots of $x^n - 1 = 0$ are contained in the series

$$1,\ a,\ a^2, \ldots\ldots a^{n-2},\ a^{n-1}.$$

(10.) We next come to the case of $x^n + 1 = 0$, all whose roots are impossible except one when n is odd. For if n be even it is manifest that every real quantity, positive or negative, when substituted for x gives a positive result, and therefore cannot be a root; and when n is odd, expelling the factor $x + 1$, the depressed equation is

$$x^{n-1} - x^{n-2} + x^{n-3} - \ldots\ldots - x + 1 = 0,$$

which cannot have a negative root (Art. 3), and it cannot have a positive root because the proposed cannot have a positive root, therefore all its roots are imaginary.

(11.) To solve the equation $x^n + 1 = 0$.

As before we may assume

$$x = \cos\theta \pm \sqrt{-1}\,\sin\theta,$$

$$\therefore\ x^n = \cos n\theta \pm \sqrt{-1}\,\sin n\theta,$$

hence all values of θ which satisfy the condition

$$\cos n\theta \pm \sqrt{-1}\,\sin n\theta = -1$$

will give values of x which are roots of the proposed; hence we must separately have $\sin n\theta = 0$, $\cos n\theta = -1$; therefore $n\theta$ must be an odd multiple of π, $= (2\lambda + 1)\pi$ suppose, where λ is any integer whatever. Hence all values of x comprised in the formula

$$x = \cos\frac{(2\lambda + 1)\pi}{n} \pm \sqrt{-1}\,\sin\frac{(2\lambda + 1)\pi}{n}$$

are roots of $x^n + 1 = 0$, or are n^{th} roots of negative unity.

Moreover this formula will give for x, n different values and no more.

For, taking λ from 0 to $\frac{1}{2}(n - 1)$ or $\frac{1}{2}n - 1$, according as n is odd or even, we find, in the former case, $\frac{1}{2}(n - 1)$ pairs of imaginary values corresponding to values of λ from 0 to $\frac{1}{2}(n - 1) - 1$, and one real value -1 for $\lambda = \frac{1}{2}(n - 1)$, or n values on the whole; and, in the latter case, we find $\frac{1}{2}n$ pairs of imaginary values corresponding to values of λ from 0 to $\frac{1}{2}n - 1$, or n values on the whole. And all these imaginary roots are different because the angles involved in them

$$\frac{\pi}{n},\ \frac{3\pi}{n},\ \frac{5\pi}{n}\ \ldots\ldots\ \frac{(n-2)\pi}{n} \text{ or } \frac{(n-1)\pi}{n}\ \ldots \quad (3)$$

are all different and less than π. And the above-mentioned n values are all which the formula can give for x. For if we

take negative multiples of π, the values of x are the same as if those multiples were positive; and if we take λ = or $> n$, the effect is to add a multiple of 2π to one of the angles (3), which alters neither the cosine nor sine. If $\lambda = n - 1$

$$x = \cos\frac{(2n-1)\pi}{n} \pm \sqrt{-1}\sin\frac{(2n-1)\pi}{n} = \cos\frac{-\pi}{n} \pm \sqrt{-1}\sin\frac{-\pi}{n}$$

$$= \cos\frac{\pi}{n} \mp \sqrt{-1}\sin\frac{\pi}{n}$$

the same as when $\lambda = 0$;
and if $\lambda = n - 1 - m$

$$x = \cos\frac{(2n-2m-1)\pi}{n} \pm \sqrt{-1}\sin\frac{(2n-2m-1)\pi}{n}$$

$$= \cos\frac{(2m+1)\pi}{n} \mp \sqrt{-1}\sin\frac{(2m+1)\pi}{n}$$

the same as when $\lambda = m$; so that values of λ, equally distant from 0 and $n - 1$, give the same values of x, and therefore we can get no new values by taking $\lambda > \frac{1}{2}(n-1)$.

Therefore the formula can never assume any other values than the n different ones resulting from taking λ from 0 to $\frac{1}{2}(n-1)$ or $\frac{1}{2}n - 1$, according as n is odd or even; since therefore the formula

$$x = \cos\frac{(2\lambda+1)\pi}{n} \pm \sqrt{-1}\sin\frac{(2\lambda+1)\pi}{n}$$

equally expresses all the roots of $x^n + 1 = 0$, and no other quantities, it is the complete solution of that equation.

(12.) As in the former case it may be shewn that if a denote the first imaginary root $\cos\frac{\pi}{n} + \sqrt{-1}\sin\frac{\pi}{n}$, all the imaginary roots may be represented by

$$a,\ a^3,\ a^5, \ldots\ a^{n-2} \text{ or } a^{n-1},$$

$$\frac{1}{a},\ \frac{1}{a^3},\ \frac{1}{a^5}, \ldots\ \frac{1}{a^{n-2}} \text{ or } \frac{1}{a^{n-1}},$$

or since $a^n = -1$, and therefore $a^{2n} = 1$, the lower line may be replaced by

$$a^{2n-1}, \quad a^{2n-2} \ldots\ldots a^{n+2} \text{ or } a^{n+1};$$

therefore, since $a^n = -1$ when n is odd, all the roots of $x^n + 1 = 0$ are contained in the series

$$a, \quad a^3, \ldots\ldots a^{2n-3}, \quad a^{2n-1}.$$

It may be observed that the case of $x^n + 1 = 0$ (n odd) might have been reduced to that of $y^n - 1 = 0$ by making $x = -y$.

We shall now give the resolution of $x^n \pm 1$ into its factors.

(13.) To resolve $x^n - 1$ into its factors.

Put $x^n - 1 = 0$;

$\therefore\ x^n = 1 = \cos 2\lambda\pi \pm \sqrt{-1} \sin 2\lambda\pi$, λ being any integer;

$$\therefore\ x = \cos\frac{2\lambda\pi}{n} \pm \sqrt{-1}\sin\frac{2\lambda\pi}{n},$$

a pair of values (except when $2\lambda = 0$ or any multiple of n when there will be only one value) to which corresponds the quadratic factor

$$x^2 - 2x\cos\frac{2\lambda\pi}{n} + 1,$$

where λ begins from 1.

First, let n be even, then $+1$ and -1 are roots, and $x^2 - 1$ a factor; and, by taking λ from 0 to $\frac{1}{2}n - 1$, we obtain the other quadratic factors,

$$\therefore x^n - 1 = (x^2 - 1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots$$

$$\left(x^2 - 2x\cos\frac{(n-2)\pi}{n} + 1\right).$$

If we take λ greater than $\frac{1}{2}n - 1$, or less than 1, the factors recur.

Secondly, let n be odd, then $+\ 1$ is a root, and $x - 1$ a simple factor; and by taking λ from 1 to $\frac{1}{2}(n - 1)$ we obtain all the quadratic factors,

$$\therefore x^n - 1 = (x-1)\left(x^2 - 2x\cos\frac{2\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{n} + 1\right)\ldots$$
$$\ldots\left(x^2 - 2x\cos\frac{(n-1)\pi}{n} + 1\right).$$

(14.) To resolve $x^n + 1$ into its factors.

Let $x^n + 1 = 0$.

$\therefore\ x^n = -1 = \cos(2\lambda + 1)\pi \pm \sqrt{-1}\sin(2\lambda + 1)\pi$,

λ being any integer;

$$\therefore\ x = \cos\frac{(2\lambda + 1)\pi}{n} \pm \sqrt{-1}\sin\frac{(2\lambda + 1)\pi}{n},$$

a pair of values (except when $2\lambda + 1$ is any multiple of n, when there will be only one value) to which corresponds the quadratic factor

$$x^2 - 2x\cos\frac{(2\lambda + 1)\pi}{n} + 1,$$

where λ begins from zero.

First, let n be even, then taking λ from 0 to $\frac{1}{2}n - 1$, we have all the quadratic factors,

$$\therefore\ x^n + 1 = \left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots$$
$$\ldots\left(x^2 - 2x\cos\frac{(n-1)\pi}{n} + 1\right).$$

Secondly, let n be odd, then -1 is a root, and $x + 1$ a simple factor; and by taking λ from 0 to $\frac{1}{2}(n - 1) - 1$, we find the $\frac{1}{2}(n - 1)$ quadratic factors,

$$\therefore x^n + 1 = (x+1)\left(x^2 - 2x\cos\frac{\pi}{n} + 1\right)\left(x^2 - 2x\cos\frac{3\pi}{n} + 1\right)\ldots$$
$$\ldots\left(x^2 - 2x\cos\frac{(n-2)\pi}{n} + 1\right).$$

(15.) To resolve $x^{2n} - 2\cos\theta x^n + 1$ into its quadratic factors.

Solving the equation $x^{2n} - 2\cos\theta x^n + 1 = 0$, we find

$x^n = \cos\theta \pm \sqrt{-1}\sin\theta = \cos(2\lambda\pi + \theta) \pm \sqrt{-1}\sin(2\lambda\pi + \theta)$, λ being any positive integer;

$$\therefore x = \cos\frac{2\lambda\pi + \theta}{n} \pm \sqrt{-1}\sin\frac{2\lambda\pi + \theta}{n},$$

a pair of values, to which corresponds the quadratic factor

$$x^2 - 2x\cos\frac{2\lambda\pi + \theta}{n} + 1;$$

and by taking λ from 0 to $n - 1$, we shall obtain the n quadratic factors required;

$$\therefore x^{2n} - 2\cos\theta x^n + 1 = (x^2 - 2x\cos\frac{\theta}{n} + 1)(x^2 - 2x\cos\frac{2\pi + \theta}{n} + 1)$$
$$(x^2 - 2x\cos\frac{4\pi + \theta}{n} + 1)\dots(x^2 - 2x\cos\frac{2(n-1)\pi + \theta}{n} + 1).$$

When n is even, since

$$\cos\frac{2\lambda\pi + \theta}{n} = -\cos\left(\frac{2\lambda\pi + \theta}{n} + \pi\right) = -\cos\left(\frac{2(\lambda + \frac{1}{2}n)\pi + \theta}{n}\right),$$

it appears that the factors corresponding to λ and to $\lambda + \frac{1}{2}n$ will only differ in the sign of the second term; therefore when we have obtained the first half of the factors by taking λ from 0 to $\frac{1}{2}n - 1$, we may find the next half corresponding to values of λ from $\frac{1}{2}n$ to $n - 1$, by changing the signs of the second terms of the former.

(16). Also, since $x^{2n} - 2\cos\theta x^n + 1$ remains unaltered when we change the sign of θ, its quadratic factors may be arranged in pairs under the general form

$$x^2 - 2x\cos\frac{2\lambda\pi \pm \theta}{n} + 1,$$

where λ is to be taken from 0 to $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd; it being observed that each of the values $\lambda = 0$, $\lambda = \frac{1}{2}n$, gives only a single factor, and not a pair.

(17.) To resolve $\sin n\theta$, $\cos n\theta$, into their factors, n being any integer.

We know that these can be expressed by polynomials containing only powers of $\sin\theta$, and of which in certain cases $\cos\theta$ is also a factor; our object is to determine the factors of those polynomials.

First, suppose n odd, then

$$x^{2n}-2\cos\theta\, x^n+1 = \left(x^2 - 2x\cos\frac{\theta}{n} + 1\right)$$

$$\left(x^2 - 2x\cos\frac{2\pi\pm\theta}{n} + 1\right)\left(x^2 - 2x\cos\frac{4\pi\pm\theta}{n} + 1\right)\ldots\ldots$$

$$\ldots\ldots\left(x^2 - 2x\cos\frac{(n-1)\,\pi\pm\theta}{n} + 1\right).$$

Now make $x = 1$, and for $\frac{\theta}{n}$ write 2θ, and for $\frac{\pi}{n}$, $2a$; and take the root of both sides,

$$\therefore\ \sin n\theta = 2^{n-1}\sin\theta\sin(2a\pm\theta)\sin(4a\pm\theta)\ldots\ldots$$

$$\ldots\ldots\sin\{(n-1)\,a\pm\theta\},$$

and changing $n\theta$ into $\frac{\pi}{2} + n\theta$; that is, θ into $\theta + a$,

$$\cos n\theta = 2^{n-1}\sin(a+\theta)\sin(a-\theta)\sin(3a+\theta)\sin(3a-\theta)\ldots$$

$$\ldots\ldots\sin\{(n-2)\,a-\theta\}\sin(na+\theta).$$

or

$$\cos n\theta = 2^{n-1}\cos\theta\sin(a\pm\theta)\sin(3a\pm\theta)\ldots\sin\{(n-2)a\pm\theta\}.$$

Now transform each pair of sines by the formula $\sin(\beta+\theta)\sin(\beta-\theta) = \sin^2\beta - \sin^2\theta$, and we have the required resolutions of $\sin n\theta$ and $\cos n\theta$ into their factors (n being odd),

$$\sin n\theta = 2^{n-1}\sin\theta(\sin^2 2a-\sin^2\theta)(\sin^2 4a-\sin^2\theta)\ldots\ldots$$

$$\ldots\ldots\{\sin^2(n-1)\,a-\sin^2\theta\}$$

$$\cos n\theta = 2^{n-1}\cos\theta(\sin^2 a-\sin^2\theta)(\sin^2 3a-\sin^2\theta)\ldots\ldots$$

$$\ldots\ldots\{\sin^2(n-2)\,a-\sin^2\theta\}.$$

Similarly, when n is even, we find

$$\sin n\theta = 2^{n-1}\cos\theta\sin\theta\,(\sin^2 2a-\sin^2\theta)(\sin^2 4a-\sin^2\theta)\ \ldots$$
$$\ldots\ldots\{\sin^2(n-2)\,a-\sin^2\theta\}$$

$$\cos n\theta = 2^{n-1}(\sin^2 a-\sin^2\theta)(\sin^2 3a-\sin^2\theta)\ \ldots\ldots$$
$$\ldots\ldots\{\sin^2(n-1)\,a-\sin^2\theta\}.$$

(18.) Hence we can resolve $\sin\theta$ and $\cos\theta$ into their factors.

If we change θ into $\frac{\theta}{n}$ we have, n being odd,

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\left(\sin^2 2a-\sin^2\frac{\theta}{n}\right)\left(\sin^2 4a-\sin^2\frac{\theta}{n}\right)\ \ldots\ldots$$

therefore, making $\theta = 0$, since in that case $\dfrac{\sin\theta}{\sin\frac{\theta}{n}} = n$, we have

$$n = 2^{n-1}\sin^2 2a\sin^2 4a\ \ldots\ldots$$

$$\therefore\ \sin\theta = n\sin\frac{\theta}{n}\left(1-\frac{\sin^2\frac{\theta}{n}}{\sin^2 2a}\right)\left(1-\frac{\sin^2\frac{\theta}{n}}{\sin^2 4a}\right)\ \ldots\ldots$$

Now make $n = \infty$, and observe that $a = \frac{\pi}{2n}$, and

$$\therefore\ \frac{\sin\frac{\theta}{n}}{\sin 2a} = \frac{\sin\frac{\theta}{n}}{\sin\frac{\pi}{n}} = \frac{\theta}{\pi},\ \ldots\ldots$$

$$\therefore\ \sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\ldots$$

Similarly $\cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots\ldots$

The same values of $\sin\theta$ and $\cos\theta$ may of course be obtained from the formulæ for $\sin n\theta$ and $\cos n\theta$ when n is even.

(19.) If we develope these values of $\sin\theta$ and $\cos\theta$ according to ascending powers of θ, and compare the results with the common formulæ

$$\sin\theta = \theta - \frac{\theta^3}{1.2.3} + \ldots, \quad \cos\theta = 1 - \frac{\theta^2}{1.2} + \ldots$$

we find

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6},$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}.$$

(20.) By making $x = 1$ in the result of example 15, we may deduce two important formulæ: first, we have

$$2(1-\cos\theta) = 2\left(1-\cos\frac{\theta}{n}\right)2\left(1-\cos\frac{2\pi+\theta}{n}\right)$$

$$2\left(1-\cos\frac{4\pi+\theta}{n}\right)\ldots 2\left(1-\cos\frac{2(n-1)\pi+\theta}{n}\right);$$

therefore, replacing $1 - \cos\theta$ by $2\sin^2\frac{\theta}{2}, \ldots$ extracting the root of both sides, and changing θ into 2θ, we have

$$\sin\theta = 2^{n-1}\sin\frac{\theta}{n}\sin\frac{\pi+\theta}{n}\sin\frac{2\pi+\theta}{n}\ldots\ldots\sin\frac{(n-1)\pi+\theta}{n},$$

and writing $\frac{\pi}{2} + \theta$ for θ,

$$\cos\theta = 2^{n-1}\sin\frac{\pi+2\theta}{2n}\sin\frac{3\pi+2\theta}{2n}\sin\frac{5\pi+2\theta}{2n}\ldots$$

$$\ldots\sin\frac{(2n-1)\pi+2\theta}{2n}.$$

(21.) If in the expression $x^2 - 2rx\cos\phi + r^2$ we write $1 + \frac{z}{2n}$ for x, and $1 - \frac{z}{2n}$ for r, it becomes

$$\left(1+\frac{z}{2n}\right)^2 - 2\left(1-\frac{z^2}{4n^2}\right)\cos\phi + \left(1-\frac{z}{2n}\right)^2 = 2\left(1+\frac{z^2}{4n^2}\right)$$

$$-2\left(1-\frac{z^2}{4n^2}\right)\cos\phi = 4\sin^2\frac{\phi}{2}\left(1+\frac{z^2}{4n^2}\cot^2\frac{\phi}{2}\right).$$

If then $\phi = \dfrac{2\lambda\pi \pm \theta}{n}$, this is the general form of the quadratic factor of

$$\left(1 + \frac{z}{2n}\right)^{2n} - 2\left(1 - \frac{z^2}{4n^2}\right)^n \cos\theta + \left(1 - \frac{z}{2n}\right)^{2n};$$

and this quantity therefore, taking λ from 0 to $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$,

$$= 4\sin^2\frac{\theta}{2n} \cdot 4\sin^2\frac{2\pi \pm \theta}{2n} \ldots \left(1 + \frac{z^2}{4n^2}\cot^2\frac{\theta}{2n}\right)$$

$$\left(1 + \frac{z^2}{4n^2}\cot^2\frac{2\pi \pm \theta}{2n}\right) \ldots$$

Now make $n = \infty$, observing that, in that case, $\left(1 \pm \dfrac{z}{2n}\right)^{2n}$ $= e^{\pm z}$ (e denoting the base of *Napier's* system of logarithms), $2n \tan\dfrac{\theta}{2n} = \theta$, and that, by putting $z = 0$ we have

$$2\sin\frac{\theta}{2n} \cdot 2\sin\frac{2\pi \pm \theta}{2n} \ldots = 2\sin\frac{\theta}{2},$$

$$\therefore\ e^z - 2\cos\theta + e^{-z} = 4\sin^2\frac{\theta}{2}\left(1 + \frac{z^2}{\theta^2}\right)$$

$$\left(1 + \frac{z^2}{(2\pi \pm \theta)^2}\right)\left(1 + \frac{z^2}{(4\pi \pm \theta)^2}\right) \ldots$$

where both signs are to be taking. Several of the preceding results are useful in the higher branches of mathematics, especially in the Integral Calculus.

SECTION II.

ON THE TRANSFORMATION OF EQUATIONS.

23. In discovering the properties of $f(x) = 0$, and determining its roots, one method of great value is to transform it into another equation $\phi(y) = 0$, whose roots have given relations with its roots. We thus, without knowing the roots of a proposed equation, make them undergo certain changes, such as all to be increased or diminished by a given quantity, or all to be multiplied or divided by the same number, which render the determination of the roots easier, or the equation in its new form more convenient for solution.

The problem of transforming an equation is, in its most general state, to eliminate x between the equation $f(x) = 0$ $\psi(x, y) = 0$, the latter being the equation which expresses the relation which the roots of the transformed are required to have with those of the proposed equation.

At present we shall confine ourselves to a few simple cases which are necessary in the actual solution of equations, reserving the others to the chapter on Elimination..

24. To transform an equation into one whose roots are those of the proposed equation with contrary signs.

Let $a, b, c, \ldots . l$ be the roots of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots . + p_n = 0;$$

$$\therefore x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=(x-a)(x-b)(x-c)\ldots.(x-l).$$

In this identical equation change x into $-y$; then, whether n be odd or even, the result is

$$y^n-p_1y^{n-1}+p_2y^{n-2}-\ldots.\pm p_n = (y+a)(y+b)(y+c)\ldots(y+l),$$

which shews that the first member put equal to zero is an equation in y whose roots are $-a, -b, -c, \ldots \ldots -l$. Hence if the signs of the alternate terms, beginning with the second, of any complete equation be changed, the signs of all the roots are changed. Before this theorem can be applied to incomplete equations, the deficient terms must be replaced by cyphers. Thus the equation $x^4 - qx + r = 0$ or $x^4 + 0x^3 + 0x^2 - qx + r = 0$ is one whose roots differ from those of $x^4 + qx + r = 0$ only in sign.

25. To transform an equation into one whose roots are those of the proposed equation, each diminished or increased by the same given quantity.

Let $a, b, c, \ldots . l$ be the roots of the equation

$$x^n + p_1x^{n-1} + \ldots . + p_{n-1}x + p_n = 0,$$

then $x^n + p_1x^{n-1} + \ldots . + p_{n-1}x + p_n = (x-a)(x-b)$
$$(x-c)\ldots.(x-l).$$

In this identical equation change x into $y + h$,

$$\therefore (y+h)^n + p_1(y+h)^{n-1} + \ldots . + p_{n-1}(y+h) + p_n$$
$$= (y - \overline{a-h})(y - \overline{b-h}) \ldots . (y - \overline{l-h}),$$

which shews that the first member, put equal to zero, is an equation in y, whose roots are $a-h, b-h, \ldots ., l-h$;

or, expanding by the binomial theorem and arranging it according to powers of y, we have the transformed equation

$$\left.\begin{array}{l} y^n + nh \\ \quad + p_1 \end{array}\right\} y^{n-1} + \left.\begin{array}{l} \frac{n(n-1)}{2}h^2 \\ + (n-1)p_1h \\ + \quad\quad p_2 \end{array}\right\} y^{n-2} + \ldots + \left.\begin{array}{l} h^n \\ + p_1h^{n-1} \\ + p_2h^{n-2} \\ + \ldots \\ + p_{n-1}h \\ + p_n \end{array}\right\} = 0.$$

To increase on the contrary all the roots, we must change x into $y - h$, and therefore we must take the odd powers of h in the above equation with a contrary sign.

26. If we arrange the transformed equation

$$(y+h)^n + p_1(y+h)^{n-1} + p_2(y+h)^{n-2} + \ldots + p_{n-1}(y+h) + p_n = 0$$

according to ascending powers of y, we shall see the law of formation of its coefficients more distinctly; for we then have

$$\begin{aligned} & h^n + p_1h^{n-1} + p_2h^{n-2} + \ldots + p_{n-2}h^2 + p_{n-1}h + p_n \\ & + \{nh^{n-1} + (n-1)p_1h^{n-2} + (n-2)p_2h^{n-3} + \ldots + 2p_{n-2}h + p_{n-1}\}y \\ & + \{n(n-1)h^{n-2} + (n-1)(n-2)p_1h^{n-3} + \ldots + 2p_{n-2}\}\frac{y^2}{\lfloor 2} \\ & + \ldots\ldots\ldots\ldots\ldots\ldots \\ & + \{n(n-1)\ldots 3.2\}\frac{y^n}{\lfloor n} = 0, \end{aligned}$$

where $\lfloor n$ denotes $1.2.3 \ldots n$.

The first coefficient is the original polynomial with h instead of x, and will therefore be represented by $f(h)$; the second coefficient is derived from the first by multiplying every term in $f(h)$ by the index of that power of h which it involves and diminishing the index by unity, and may be represented by $f'(h)$; the third is found from the second, by repeating the

same process upon $f'(h)$ or performing it twice upon $f(h)$, and may therefore be represented by $f''(h)$; and in like manner all the other coefficients, being formed by the same uniform law, may be represented by $f'''(h), \ldots\ldots f^{n-1}(h)$; therefore the transformed equation arranged according to ascending powers of y, is

$$f(h)+f'(h)y+f''(h)\frac{y^2}{1.2}+f'''(h)\frac{y^3}{\lfloor 3}+\ldots+f^{n-1}(h)\frac{y^{n-1}}{\lfloor n-1}+y^n=0.$$

27. Hence it follows that if in $f(x)$ we change x into $x+h$, the result arranged according to powers of h, is

$$f(x+h)=fx+f'(x)h+f''(x)\frac{h^2}{1.2}+\ldots+f^{n-1}(x)\frac{h^{n-1}}{\lfloor n-1}+f^n(x)\frac{h^n}{\lfloor n}.$$

$f'(x), f''(x), \ldots . f^{n-1}(x)$ being derived from $f(x)$ according to the laws explained above; they are called derived functions relative to the given function $f(x)$. Those who are acquainted with the Differential Calculus know that this result can be immediately obtained from *Taylor's* theorem; and that $f'(x)$ $f''(x), \ldots .$ are the first, second, ... differential coefficients of $f(x)$, which all vanish after the n^{th}, $f(x)$ being a rational integral function of the n^{th} degree.

28. Hence to increase or diminish the roots of a proposed equation $f(x) = 0$, by a given quantity h, we must write down $f(x)$ and all the derived functions, $f(x), f'(x), f''(x) \ldots\ldots$ $f^{n-1}(x)$, and substitute in them $-h$ or $+h$ for x according as the roots are to be increased or diminished; the resulting quantities are the coefficients of the transformed equation.

Ex. To transform $x^5 + 5x^4 + x^3 - 16x^2 - 20x - 16 = 0$, into one whose roots shall be the same except that each is increased by unity.

Therefore $y = x + 1$ or $x = y - 1$;

therefore the transformed equation is

$$f(-1)+f'(-1)y+f''(-1)\frac{y^2}{2}+f^3(-1)\frac{y^3}{\underline{3}}+f^4(-1)\frac{y^4}{\underline{|4}}+y^5=0;$$

but $f(x) = x^5+5x^4+x^3-16x^2-20x-16 \qquad f(-1) = -9$

$f'(x) = 5x^4+20x^3+3x^2-32x-20 \qquad f'(-1) = 0$

$f''(x) = 20x^3+60x^2+6x-32 \qquad f''(-1) = 2$

$f^3(x) = 60x^2+120x+6 \qquad f^3(-1) = -54$

$f^4(x) = 120x+120 \qquad f^4(-1) = 0;$

$$\therefore -9+2.\frac{y^2}{2}-54.\frac{y^3}{6}+y^5 = 0$$

$$\text{or } y^5 - 9y^3 + y^2 - 9 = 0,$$

the roots of which were found p. 15.

29. One use of this transformation is to take away any term of an equation, by which means it is sometimes reduced to a form more convenient for solution, as in the preceding Ex.

Thus to transform $f(x) = 0$ into one which shall want the second term, we must have $nh + p_1 = 0$, or $h = -\frac{p_1}{n}$, and therefore $x = y - \frac{p_1}{n}$; i. e. the roots must be increased by $\frac{p_1}{n}$, (p_1 being the coefficient of the second term with its proper sign and n the degree of the equation) at which we may arrive immediately by observing that if $a, b, c, \ldots . l$ be the roots of $x^n + p_1x^{n-1} + \ldots . = 0$, and $a + h, b + h, \ldots . l + h$ those of the transformed equation $y^n + q_1 y^{n-1} + \ldots . = 0$, then $-q_1 = a + b + \ldots . l + nh = -p_1 + nh$; and if this $= 0$, then $h = \frac{p_1}{n}$, the quantity by which the roots are to be increased.

To take away the third term we must diminish the roots by a quantity h determined from the equation

$$\frac{n(n-1)}{2} h^2 + (n-1) p_1 h + p_2 = 0;$$

and in general to take away the $(r+1)^{\text{th}}$ term, we must diminish the roots by a quantity h determined from the equation

$$f^{n-r}(h) = 0, \text{ or } h^r + \frac{r}{n} p_1 h^{r-1} + \frac{r(r-1)}{n(n-1)} p_2 h^{r-2} + \ldots\ldots = 0.$$

To take away the last term we have $h^n + p_1 h^{n-1} + \ldots = 0$; i. e. we must solve the original equation. In effect the transformed equation would have one root $= 0$, and therefore $h = x$.

Ex. To transform $x^3 - 6x^2 + 4x - 7 = 0$ into one which shall want the second term.

$$\text{Here } p_1 = -6,\ n = 3;\ \therefore\ x = y - \frac{p_1}{n} = y + 2;$$

$$\therefore\ (y+2)^3 - 6(y+2)^2 + 4(y+2) - 7 = 0,$$

$$\text{or } y^3 - 8y - 15 = 0.$$

30. To transform an equation into another of which the roots are equal to those of the proposed, each multiplied by the same given quantity.

In the identical equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_{n-1} x + p_n = (x-a)(x-b) \ldots\ldots (x-l)$

change x into $\frac{y}{m}$, and then multiply both sides by m^n,

$$\therefore\ y^n + m p_1 y^{n-1} + m^2 p_2 y^{n-2} + \ldots\ldots + m^{n-1} p_{n-1} y + m^n p_n$$
$$= (y - ma)(y - mb) \ldots\ldots (y - ml);$$

therefore the roots of

$$y^n + m p_1 y^{n-1} + m^2 p_2 y^{n-2} + \ldots\ldots + m^{n-1} p_{n-1} y + m^n p_n = 0$$

are $ma, mb, mc, \ldots\ldots ml$; and it is formed from the equation, supposed complete, whose roots are $a, b, c, \ldots\ldots l$, by multi-

plying the coefficients beginning with that of the second term by $m, m^2, m^3, \ldots m^n$.

The use of this transformation is to get rid of the coefficient of the first term; or to make the fractional coefficients of an equation disappear without affecting the first term with any coefficient except unity.

Thus if $mx^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots = 0$ have roots $a, b, c \ldots$

$$\text{then } my^n + mp_1 y^{n-1} + m^2 p_2 y^{n-2} + \ldots = 0$$

$$\text{or } y^n + p_1 y^{n-1} + mp_2 y^{n-2} + \ldots = 0$$

has roots $ma, mb, mc, \ldots$, and is of the usual form.

Also if $x^n + \frac{p_1}{q_1} x^{n-1} + \frac{p_2}{q_2} x^{n-2} + \frac{p_3}{q_3} x^{n-3} + \ldots\ldots = 0$ have roots $a, b, c \ldots$; and if m be the least common multiple of all the denominators $q_1, q_2, q_3, \ldots$, then

$$y^n + \frac{mp_1}{q_1} y^{n-1} + \frac{m^2 p_2}{q_2} y^{n-2} + \ldots = 0$$

has roots $ma, mb, mc, \ldots\ldots$, and all its coefficients are integers.

Similarly an equation may be transformed into another of which the roots are equal to those of the proposed, each divided by the same given quantity, by dividing the second, third, fourth terms, &c. (supposing the equation complete) by $m, m^2, m^3, \ldots$ respectively.

31. By taking $m =$ the least common multiple of the denominators, we do not always get the transformed equation with the least possible coefficients. All that is necessary is to determine m so that $m, m^2, m^3 \ldots$ are divisible respectively by $q_1, q_2, q_3, \ldots\ldots$; and therefore that m, m^2, m^3 contain the prime factors of $q_1, q_2, q_3 \ldots$ raised at least to as high powers as they occur in the respective denominators.

Ex. 1. $$x^4 - \frac{4}{3}x^2 - \frac{3}{8}x + \frac{5}{72} = 0.$$

The transformed equation is

$$y^4 - m\frac{4}{3}x^2 - m^2\frac{3}{2^3}x + m^3\frac{5}{3^2.2^3} = 0,$$

and the factors which the successive terms require in m are 3, 2^2, 3.2; which is satisfied by $m = 12$, and the transformed equation is

$$y^4 - 16y^2 - 54y + 120 = 0 \quad \text{where } y = 12x.$$

Ex. 2. $$x^4 - \frac{5}{6}x^3 + \frac{5}{12}x^2 - \frac{7}{150}x - \frac{13}{900} = 0.$$

The transformed equation is

$$y^4 - 25y^3 + 375y^2 - 1260y - 11700 = 0 \quad \text{where } y = 30x.$$

32. To transform an equation into one whose roots are the reciprocals of the roots of the proposed equation.

If in the identical equation $x^n + p_1 x^{n-1} + \ldots + p_{n-1}x + p_n$ $= (x-a)(x-b)\ldots\ldots(x-l)$ we change x into $\frac{1}{y}$, we have

$$\frac{1}{y^n} + \frac{p_1}{y^{n-1}} + \ldots\ldots + \frac{p_{n-1}}{y} + p_n = \left(\frac{1}{y} - a\right)\left(\frac{1}{y} - b\right)\ldots\left(\frac{1}{y} - l\right)$$

$$= -\frac{a}{y}\left(y - \frac{1}{a}\right) . -\frac{b}{y}\left(y - \frac{1}{b}\right)\ldots\ldots$$

$$= \frac{p_n}{y^n}\left(y - \frac{1}{a}\right)\left(y - \frac{1}{b}\right)\ldots\left(y - \frac{1}{l}\right)$$

or $$y^n + \frac{p_{n-1}}{p_n}y^{n-1} + \frac{p_{n-2}}{p_n}y^{n-3} + \ldots\ldots + \frac{p_1}{p_n}y + \frac{1}{p_n}$$

$$= \left(y - \frac{1}{a}\right)\left(y - \frac{1}{b}\right)\ldots\ldots\left(y - \frac{1}{l}\right)$$

which shews that if we write $\frac{1}{y}$ for x, and then multiply by y^n, the resulting expression put $= 0$, has roots $\frac{1}{a}, \frac{1}{b} \dots \frac{1}{l}$.

33. This transformation fails if the transformed equation be identical with the original one; that is, if the coefficients be such that

$$p_{n-1} = p_n p_1, \ p_{n-2} = p_n p_2, \dots, 1 = p_n p_n.$$

Hence $p_n = \pm 1$, and according as we take the upper or lower sign, we have

$$p_{n-1} = p_1, \qquad p_{n-2} = p_2, \dots$$

$$\text{or } p_{n-1} = -p_1, \ p_{n-2} = -p_2, \dots;$$

that is, the coefficients of corresponding terms taken from the beginning and end must be equal and of the same signs, or equal and of contrary signs; only it must be observed that if the equation be of an even number of dimensions $2r$, there will be a middle term $p_r x^r$, and we shall have $p_r = p_r p_{2r}$ which, for $p_{2r} = -1$, gives $p_r = -p_r$ or $p_r = 0$; so that when the equation is of an even degree and the corresponding coefficients have contrary signs, there must be no middle term.

It is easy to see that when these conditions are satisfied, the equation remains the same when $\frac{1}{x}$ is substituted for x, but when the corresponding coefficients have contrary signs it will be necessary, after the substitution, to change the signs of all the terms; the above investigation shews that these are the only conditions under which an equation can have the property. Equations of this sort, that is, which remain the same when x is changed into $\frac{1}{x}$, are called reciprocal equations.

34. Every reciprocal equation will have its roots in pairs $a, \frac{1}{a}, b, \frac{1}{b} \ldots$; but when the degree is odd, there will besides be a root $+1$ or -1 according as the last term is negative or positive; and when the degree is even with the last term negative, there will be two roots $+1$ and -1.

For if a be a root of the proposed equation, $\frac{1}{a}$ will be a root of the transformed equation; but the transformed equation coincides with the proposed, therefore $\frac{1}{a}$ will be a root of the proposed equation, and so on for every other root. Also since 1 and -1 are the same as their reciprocals, each of these may enter any even number of times.

Again, the reciprocal equation of an odd degree may be written, collecting the terms from the beginning and end,

$$x^n \pm 1 + p_1 x (x^{n-2} \pm 1) + \ldots = 0;$$

and since every term is divisible by $x \pm 1$, it will have a root $+1$ or -1 according as its last term is negative or positive.

Also the reciprocal equation of an even degree with its last term negative may be written

$$x^n - 1 + p_1 x (x^{n-2} - 1) + \ldots = 0,$$

which is divisible by $x^2 - 1$; therefore it has two roots $+1$ and -1.

In both cases when the factor $x \pm 1$ or $x^2 - 1$ is expelled, the equation is reduced to a reciprocal equation of an even degree with its last term positive, which may therefore be taken as the standard form of reciprocal equations.

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35. Various transformations may be effected by particular artifices; we shall give one or two instances where the results will be useful to us in the sequel.

(1.) To transform an equation into one whose roots are the squares of the roots of the proposed.

If $x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots + p_{n-1}x + p_n$
$$= (x-a)(x-b)\ldots\ldots(x-l),$$
then $x^n - p_1x^{n-1} + p_2x^{n-2} - p_3x^{n-3} + \ldots \pm p_{n-1}x \mp p_n$
$$= (x+a)(x+b)\ldots\ldots(x+l);$$
therefore, multiplying these equations together, we have
$$(x^n + p_2x^{n-2} + p_4x^{n-4} + \ldots)^2 - (p_1x^{n-1} + p_3x^{n-3} + \ldots)^2$$
$$= (x^2 - a^2)(x^2 - b^2)\ldots\ldots(x^2 - l^2),$$
but the first member is
$$x^{2n} + 2p_2x^{2n-2} + (2p_4 + p_2^2)x^{2n-4} + \ldots\ldots$$
$$- (p_1^2x^{2n-2} + 2p_1p_3x^{2n-4} + \ldots\ldots);$$
therefore, replacing x^2 by y, we have
$$y^n + (2p_2 - p_1^2)y^{n-1} + (p_2^2 - 2p_1p_3 + 2p_4)y^{n-2} + \ldots\ldots$$
$$= (y - a^2)(y - b^2)\ldots\ldots(y - l^2),$$
hence the transformed equation, whose roots are $a^2, b^2, \ldots$, is
$$y^n + (2p_2 - p_1^2)y^{n-1} + (p_2^2 - 2p_1p_3 + 2p_4)y^{n-2} + \ldots = 0.$$

(2.) To transform the equation $x^3 + qx + r = 0$ into one whose roots are the squares of the differences of its roots.

Let the roots of $x^3 + qx + r = 0$ be a, b, c;
$$\therefore 0 = a + b + c, \quad q = ab + ac + bc, \quad -r = abc,$$
$$\text{and } a^2 + b^2 + c^2 = -2q.$$

Since one root of the transformed equation is
$$(a-b)^2 = a^2 + b^2 + c^2 - c^2 - \frac{2abc}{c} = -2q - c^2 + \frac{2r}{c},$$
if we assume $y = -2q - x^2 + \dfrac{2r}{x}$, then when x assumes its three values, y becomes equal to the three roots of the trans-

formed equation; therefore the required equation will result from eliminating x between the proposed and

$$x^3 + (y + 2q)\,x - 2r = 0,$$

subtracting this from the proposed, we have

$$(y + q)\,x - 3r = 0 \text{ or } x = \frac{3r}{y + q},$$

and if we substitute this value, and reduce, we obtain the transformed equation

$$y^3 + 6qy^2 + 9q^2y + 108\left(\frac{r^2}{4} + \frac{q^3}{27}\right) = 0.$$

Hence if $\frac{r^2}{4} + \frac{q^3}{27}$ is positive, the transformed has a real negative root (Art. 10,) and therefore the proposed equation must have a pair of imaginary roots; since it is only when two roots are imaginary and conjugate to one another that the square of their difference can be negative.

If $\frac{r^2}{4} + \frac{q^3}{27} = 0$, then one value of y is zero; and therefore the proposed has a pair of equal roots.

If $\frac{r^2}{4} + \frac{q^3}{27}$ is negative (and therefore q an essentially negative quantity) the transformed equation cannot have a negative root (Art. 3); and therefore the proposed has all its roots real.

SECTION III.

ON THE LIMITS OF THE ROOTS OF EQUATIONS.

36. THE limits of any group of roots of an equation are two quantities between which the whole group lies; thus $+\infty$ and 0 are limits of the positive roots of every equation, and 0 and $-\infty$ of the negative roots. But in practice we are required to assign much closer limits than these, usually the two consecutive whole numbers between which each root lies, so that the inferior limit is the integral part of the included root. This may be effected without knowing any of the roots of the equation, as will be seen in the following propositions. The roots spoken of in this section are the real roots.

37. Quantities between which the real roots of an equation taken in order lie, when substituted successively for the unknown quantity, give results alternately positive and negative.

Let the real roots arranged in order of magnitude be $a, b, c, \ldots . l$, so that a is greater than b, b greater than c, $\ldots .$; the negative roots, if there be any, coming at the end of the series, and that being the least whose numerical value (neglecting the sign) is greatest; then if $f(x) = 0$ be the equation,

$$f(x) = (x - a)(x - b)(x - c) \ldots (x - l) \,.\, \phi(x),$$

where $\phi(x)$ is a polynomial that remains positive whatever values be substituted in it for x, (Art. 14.) Then if we substitute for x a quantity α greater than a, the result $f(\alpha)$ is positive because every one of its factors is so; if we substitute a quantity β between a and b, the result $f(\beta)$ is negative because the first factor is negative and the rest positive. Again, a quantity between b and c renders the whole positive, because the two first factors are negative and the rest positive. Thus quantities between which the roots taken in order lie, when substituted for x, give results alternately positive and negative.

38. Again, suppose that $a, b, c, \ldots . l$ are all the roots of $f(x) = 0$, which lie between two numbers α and β, of which α is the lesser, and that $\phi(x) = 0$ is the equation containing the remaining roots; then substituting α and β successively for x, and dividing one result by the other,

$$\frac{f(\alpha)}{f(\beta)} = \frac{(\alpha - a)(\alpha - b)\ldots(\alpha - l)}{(\beta - a)(\beta - b)\ldots(\beta - l)} \cdot \frac{\phi(\alpha)}{\phi(\beta)}.$$

Now all the factors in the numerator are negative, and all in the denominator positive, also $\phi(\alpha)$, $\phi(\beta)$ must have the same sign, since $\phi(x) = 0$ has no root between α and β; therefore $f(\alpha)$, $f(\beta)$ have different or the same signs according as the number of factors $\alpha - a$, $\alpha - b, \ldots$ is odd or even. Hence if two numbers, when substituted for x, give results with different signs, then one, three, or some odd number of roots lies between them; if they give results with the same sign, then two, four, or some even number of roots lies between them, or none at all.

39. If a number α can be found such that α and every greater number, when substituted for x, gives a positive result, then α is greater than the greatest root, and is called a superior

limit of the roots; for if there could be a root greater than a, then some number greater than a would give a negative result, which is contrary to the supposition. Similarly, if a number β can be found such that β and every smaller number when substituted for x, gives a result with a permanent sign, that is, constantly positive or constantly negative, according as the degree of the equation is even or odd, β is less than the least root, and is called an inferior limit of the roots.

40. If as many quantities can be found which substituted for x in $f(x)$ give results alternately positive and negative as the equation has dimensions, it is plain that the odd number of roots which lies between each adjacent two of the quantities, cannot exceed one. But as it is seldom the case that so many can be found, the next point to be determined is, whether all the real roots that exist have been discovered; this enquiry will obviously be narrowed if we find the limits beyond which the quantities, successively substituted for the purpose of separating the roots, need not extend, that is, the superior and inferior limits of the positive and negative roots; the principal methods of doing this are the following.

41. All the roots of an equation lie between $p + 1$ and $-(p + 1)$, p being the greatest coefficient without regard to sign.

For it is proved (Art. 9) that $p + 1$ and every greater number, when substituted for x, gives a positive result, therefore $p + 1$ is greater than the greatest root; also that $-(p + 1)$ and every greater negative number gives a result with a permanent sign, that is, constantly positive or constantly negative, according as the degree of the equation is even or odd, therefore $-(p+1)$ is less than the least root.

42. The greatest negative coefficient increased by unity is a superior limit of the positive roots of an equation.

Let $-p$ be the greatest negative coefficient; then any value of x which makes

$$x^n - p(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1) \text{ positive,}$$

$$\text{or } x^n > p(x^{n-1} + x^{n-2} + \ldots + x^2 + x + 1) > p\,\frac{x^n - 1}{x - 1},$$

will, *a fortiori*, make

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1}x + p_n \text{ positive,}$$

or make $f(x)$ positive, because in the latter case there will generally be fewer terms to be taken away from x^n, and of these not one is greater than the corresponding term in the former case.

Now the inequality $x^n > p\,\dfrac{x^n - 1}{x - 1}$ is satisfied if

$$x^n = \text{or} > x^n \frac{p}{x - 1} \text{ or } x - 1 = \text{or} > p \text{ or } x = \text{or} > p + 1.$$

Since therefore $p + 1$ and every greater number, when substituted for x, will make $f(x)$ positive, the greatest negative coefficient increased by unity is a superior limit of the positive roots.

This result, as is easily seen, is included in that of the preceding article; for if all the coefficients were negative, the substitution of the greatest of them and of every greater quantity would give a positive result; therefore, *a fortiori*, the result will be positive if some of the coefficients be positive; the limit however here determined will usually be less than that in the former article, and never greater.

43. In any equation if $p_r x^{n-r}$ be the first term which is negative, and $-p$ the greatest negative coefficient, $1 + \sqrt[r]{p}$ is a superior limit of the positive roots.

Any value of x which makes

$$x^n > p\,(x^{n-r} + x^{n-r-1} + \ldots + x + 1) > p\,\frac{x^{n-r+1} - 1}{x - 1},$$

will of course make $x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots.$ positive.

Now the inequality $x^n > p\,\dfrac{x^{n-r+1} - 1}{x - 1}$ is satisfied if

$$x^n > p\,\frac{x^{n-r+1}}{x-1} \text{ or } x^{r-1}(x-1) > p \text{ or } (x-1)^{r-1}(x-1) = \text{or} > p$$

$$\text{or } (x - 1)^r = \text{or} > p \text{ or } x = \text{or} > 1 + \sqrt[r]{p}.$$

Since therefore $1 + \sqrt[r]{p}$ and every greater number gives a positive result, $1 + \sqrt[r]{p}$ is a superior limit. This method may be employed when the first term is followed by one or more positive terms.

Ex. $$x^4 + 11x^2 - 25x - 61 = 0.$$

Here $r = 3$, and the limit of the positive roots

$$= 1 + \sqrt[3]{61} = 5.$$

44. If each negative coefficient, taken positively, be divided by the sum of all the positive coefficients which precede it, the greatest of the fractions thus formed, increased by unity, is a superior limit of the positive roots.

Let the equation be

$$x^n + p_1x^{n-1} + p_2x^{n-2} + (-p_3)x^{n-3} + \ldots + (-p_r)x^{n-r} + \ldots + p_n = 0;$$

then, since (Art. 5)

$$p_mx^m = p_m(x - 1)\,(x^{m-1} + x^{m-2} + \ldots + x + 1) + p_m,$$

if we transform every positive term by this formula, and leave the negative terms in their original form, we shall have

$$\begin{aligned} 0 = (x-1)x^{n-1} + (x-1)x^{n-2} + (x-1)x^{n-3} + \ldots + x - 1 + 1 \\ + p_1(x-1)x^{n-2} + p_1(x-1)x^{n-3} + \ldots + p_1(x-1) + p_1 \\ + p_2(x-1)x^{n-3} + \ldots + p_2(x-1) + p_2 \\ - \;p_3 \quad x^{n-3} \qquad\qquad\qquad \\ + \ldots\ldots\ldots\ldots \end{aligned}$$

Now if x be taken such that every term is positive, that value will be the superior limit required; in the terms where no negative coefficient enters it is sufficient to have $x > 1$; in the other terms, each of which involves a negative coefficient, we must have

$$(1+p_1+p_2)(x-1) > p_3,\ (1+p_1+p_2+\dots+p_{r-1})(x-1) > p_r \dots$$

$$\text{or } x > \frac{p_3}{1+p_1+p_2}+1;\ x > \frac{p_r}{1+p_1+p_2+\dots+p_{r-1}}+1, \dots$$

If then x be taken equal to the greatest of these fractions increased by unity, this value, and every greater value, will make $f(x)$ positive, and therefore will be a superior limit of the positive roots. This method gives a limit easily calculated, and generally not far from the truth.

Ex. $\quad 4x^5 - 8x^4 + 23x^3 + 105x^2 - 80x + 3 = 0.$

The fractions are $\frac{8}{4}$ and $\frac{80}{4+23+105}$, and $\frac{8}{4} > \frac{80}{132}$;

therefore 3 is a superior limit.

45. The form of the equation will often suggest artifices, by means of which closer limits may be determined than by any of the preceding methods; thus, writing the above equation under the form

$$4x^4(x-2)+23x^3+105x\left(x-\frac{16}{21}\right)+3=0,$$

we see that $x =$ or > 2 gives a positive result, therefore 2 is a superior limit. Similarly, by writing the example of Art. 43 under the form

$$x(x^3-25)+11\left(x^2-\frac{61}{11}\right)=0,$$

we see that 3 is a superior limit.

46. To find an inferior limit of the positive roots, we must transform the equation into one whose roots are the reciprocals of the roots of the former; and the superior limit of the roots of the transformed equation found by the preceding methods will be the quantity required. Hence if p_r denote the greatest coefficient of a contrary sign to the last term p_n, an inferior limit of the positive roots is $\frac{p_n}{p_n + p_r}$. For the transformed equation will be (Art. 32)

$$y^n + \frac{p_{n-1}}{p_n} y^{n-1} + \ldots + \frac{p_r}{p_n} y^r + \ldots . + \frac{1}{p_n} = 0,$$

of which $\frac{p_r}{p_n}$ is the greatest negative coefficient; therefore $\frac{p_r}{p_n} + 1$ is a superior limit of its roots; and consequently $\frac{p_n}{p_r + p_n}$ an inferior limit of the positive roots of the proposed equation.

47. To find superior and inferior limits of the negative roots, we must transform the equation into one whose roots are those of the former with contrary signs (Art. 24); and if a, β, be limits, found as above, of the positive roots of this equation, then $-a$ and $-\beta$ will be limits of the negative roots of the proposed equation.

48. The limits however deduced by any of the preceding methods seldom approach very near to the roots; the tentative method, depending upon the following proposition, will furnish us with limits which lie much nearer to them.

Every number which, written for x, makes $f(x)$ and all its derived functions positive, is a superior limit of the positive roots.

For if we diminish the roots $a, b, c, \ldots .$ of $f(x) = 0$ by

h, that is, (Art. 25) substitute $y + h$ for x, the result is $f(y + h) = 0$, or

$$f(h) + f'(h)\frac{y}{1} + f''(h)\frac{y^2}{1.2} + \ldots. + f^{n-1}(h)\frac{y^{n-1}}{\lfloor n-1} + y^n = 0$$

Now if we give such a value to h that all the coefficients of this equation are positive, then every value of y is negative; that is, all the quantities $a - h$, $b - h$, $c - h$, ... are negative, and therefore h is greater than the greatest of the quantities a, b, c, ..., or a superior limit of the roots of the proposed equation.

Ex. To find a superior limit of the roots of

$$x^3 - 5x^2 + 7x - 1 = 0.$$

The transformed equation, putting $y + h$ for x, is

$$(h^3-5h^2+7h-1) + (3h^2-10h+7)y + (6h-10)\frac{y^2}{2} + y^3 = 0;$$

in which, if 3 be put for h, all the coefficients are positive, therefore 3 is a superior limit of the positive roots.

This method of determining by trial what value of x will make $f(x)$ and all its derived functions positive, was proposed by *Newton*.

49. If a series of quantities be substituted for x in $f(x)$, then between every two which give results with different signs an odd number of roots of $f(x) = 0$ is situated; and between every two which give results with the same signs an even number is situated, or none at all; but we cannot assure ourselves that in the former case the number does not exceed unity, or that in the latter it is zero, and that consequently the number and situation of all the real roots is ascertained, unless the difference between the quantities successively substituted be less than the least difference between the roots of the proposed equation; since, if it were greater, it is evident that

more than one root might be intercepted by two of the quantities giving results with different signs, and that two roots instead of none might be intercepted by two of the quantities giving results with the same sign; and in both cases roots would pass undiscovered. We must therefore first find a limit less than the least difference of the roots; this may be done by transforming (as we have already shewn for a cubic, and shall hereafter shew generally) the equation into one whose roots are the squares of the differences of the roots of the proposed equation. Then if we find a limit k less than the least positive root of the transformed equation, $\sqrt{k}$ will be less than the least difference of the roots of the proposed equation; and if we substitute successively for x the numbers s, $s - \sqrt{k}$, $s - 2\sqrt{k}$, ... (s being a superior limit of the roots of the proposed) till we come to a superior limit of the negative roots, we are sure that no two real roots lying between the numbers substituted have escaped us, and that every change of signs in the results of the substitutions indicates only one real root. Hence the number of real roots will be known (for it will exactly equal the number of changes) as well as the interval in which each of them is contained. This method of determining the number and situation of the real roots of an equation was first proposed by *Waring;* it is however of no practical use for equations exceeding the fourth degree, on account of the great labour of forming the equation of differences for equationsof a higher order.

Ex. $x^3 - 7x + 7 = 0$. The numbers 1 and 2 give each a positive result, but yet two roots lie between them. The equation whose roots are the squares of the differences is $y^3 - 42y^2 + 441y - 49 = 0$, an inferior limit of the positive roots of which is $\frac{1}{9}$, because, putting $y = \frac{1}{z}$, it may be transformed into $z^3(z - 9) + \frac{6}{7}\left(z - \frac{1}{42}\right) = 0$; therefore $\frac{1}{3}$ is less

than the least difference of the roots of $x^3 - 7x + 7 = 0$, and substituting $2, \frac{5}{3}, \frac{4}{3}$, the results are $+$, $-$, $+$; hence one value of x lies between 2 and $\frac{5}{3}$, and one between $\frac{5}{3}$ and $\frac{4}{3}$; and similarly we find the negative root, which necessarily exists, to lie between 3 and $3\frac{1}{3}$.

50. If an equation have only one change of signs, it can only have one positive root.

Since the equation has only one change of signs, it will have one or more positive terms, and all the rest will be negative; therefore it will necessarily have a positive root a; let sx^r be the last positive term, and let the equation be divided by x^r, and it will be

$$x^{n-r} + px^{n-r-1} + \ldots + s - \left(\frac{t}{x} + \ldots + \frac{u}{x^r}\right) = 0,$$

then when $x = a$ the two parts become equal, but if $x > a$, the first part increases and the second diminishes; and if $x < a$ (continuing positive) the first part diminishes and the second increases; therefore it is impossible that for any positive value except $x = a$ the two parts should be equal, or that the equation should have more than one positive root.

51. The first derived function $f'(x)$ put equal to zero, gives an equation whose roots have remarkable relations with the roots of $f(x) = 0$; these relations we now proceed to demonstrate, as well as to draw several important conclusions from them, beginning with the following proposition.

An odd number of the roots of the equation

$$f'(x) = nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \ldots + p_{n-1} = 0$$

lies between each adjacent two of the roots of $f(x) = 0$.

Let the real roots of $f(x) = 0$ arranged in order of magnitude be $a, b, c, \ldots . l$, in number $n - r$,

$$\therefore f(x) = (x-a)(x-b)(x-c)\ldots .(x-l).\phi(x)$$

where $\phi(x)$ is a polynomial of r dimensions that cannot become negative for any value of x. In this identical equation for x write $y + h$;

$$\therefore f(y+h) = (y+h-a)(y+h-b)(y+h-c)\ldots$$
$$\ldots (y+h-l).\phi(y+h),$$

$$\text{or } f(h) + f'(h)\cdot\frac{y}{1} + f''(h)\frac{y^2}{1.2} + \ldots . + y^n$$
$$= \Big\{y^{n-r} + \ldots + \{(h-b)(h-c)\ldots + (h-a)(h-c)\ldots$$
$$\ldots + (h-a)(h-b)\ldots + \ldots\}y + (h-a)(h-b)\ldots(h-l)\Big\}$$
$$\cdot\left\{\phi(h) + \phi'(h)\cdot\frac{y}{1} + \phi''(h)\frac{y^2}{1.2} + \ldots . + y^r\right\};$$

therefore, equating coefficients of y,

$$f'(h) = \{(h-b)(h-c)\ldots + (h-a)(h-c)\ldots + (h-a)(h-b)\ldots + \ldots\}$$
$$.\phi(h) + (h-a)(h-b)\ldots(h-l)\,\phi'(h),$$

(where the coefficient of $\phi(h)$ is the sum of a series of products, in forming which, each of the factors $h - a$, $h - b$, ... is left out in turn) in which equation, if $a, b, c, \ldots .$ be written for h, since the last term of the second member vanishes by these substitutions, and $\phi(h)$ is always positive, the results will have the same signs as

$$(a-b)(a-c)\ldots, (b-a)(b-c)\ldots, (c-a)(c-b)\ldots,$$

which are alternately positive and negative, since they respectively involve 0, 1, 2, ... negative factors, $a, b, c, \ldots .$ being arranged in order of magnitude. Hence an odd number of roots of $f'(h) = 0$, or of $f'(x) = nx^{n-1} + (n-1)p_1x^{n-2} + \ldots . = 0$, (since it is of no importance by what symbol we represent the unknown quantity,) lies between a and b, an odd

number between b and c, and so on; that is, an odd number of the roots of $f'(x) = 0$ lies between each adjacent two of the roots of $f(x) = 0$.

52. If the equation $f(x) = 0$ has n real roots, then

$$nx^{n-1} + (n-1)\,p_1x^{n-2} + \ldots\ldots = 0, \text{ or } f'(x) = 0$$

has $n - 1$ real roots, for one of its roots lies between each adjacent two of the roots of $f(x) = 0$; and it is therefore in this case called the limiting equation of the proposed.

53. The equation

$$n(n-1)\,x^{n-2} + (n-1)(n-2)p_1x^{n-3} + \ldots = 0, \text{ or } f''(x) = 0$$

being derived from $f'(x) = 0$ in the same manner as this latter is derived from $f(x) = 0$, will have an odd number of roots lying between each adjacent two of the roots of $f'(x) = 0$; and if all the roots of $f(x) = 0$ are real, all in $f'(x) = 0$, $f''(x) = 0, \ldots\ldots$ are real, till we arrive at a simple equation. And, in general, the equations $f'(x) = 0$, $f''(x) = 0, \ldots\ldots$ have at least as many real roots wanting one, two, $\ldots\ldots$ as $f(x) = 0$. Hence if $f(x) = 0$ has $n - r$ possible and r impossible roots, $f^m(x) = 0$ will have at least $n - r - m$ possible, and therefore (being of $n - m$ dimensions) cannot have more than r impossible roots; which shews that though $f(x) = 0$ may have fewer real roots than several of its derived equations, it has at least as many impossible roots as any one of them.

54. If we know all the real roots of $f'(x) = 0$, and substitute them in order in $f(x)$, we may find how many real roots the proposed equation contains.

For let $\alpha, \beta, \gamma, \ldots\ldots \lambda$ be the roots of $f'(x) = 0$, arranged in order of magnitude, and let

$$\infty,\ \alpha,\ \beta,\ \gamma,\ \ldots\ldots\ \lambda,\ -\infty$$

be substituted for x in $f(x)$, giving results

$$+, f(\alpha), f(\beta), \ldots f(\lambda), \pm;$$

then there can only be one root greater than α and one less than λ; for if there could be more, $f'(x) = 0$ would have a root situated between them, that is, a root $> \alpha$ or $< \lambda$, which is impossible, for $\alpha, \beta, \gamma, \ldots \lambda$ are all the real roots of $f'(x) = 0$ taken in order; also the other roots are situated singly between α and β, β and γ, ... Hence if $f(\alpha)$ be positive, there is no root greater than α; if negative, there is one root greater than α; if $f(\beta)$ have the same sign as $f(\alpha)$ there is no root between α and β, otherwise there is one root, and so on: and if $f(\lambda)$ be positive for an equation of odd dimensions, or negative for one of even dimensions, there will be one root $< \lambda$, otherwise none. It follows, therefore, that the number of real roots of $f(x) = 0$ will be exactly equal to the number of changes of sign in the results of the substitution of ∞, $\alpha, \beta, \gamma, \ldots \lambda, -\infty$ for x, and can be exactly determined whenever we can obtain a solution of $f'(x) = 0$.

Ex. 1. To determine whether $x^3 - qx + r = 0$ has all its roots possible.

The limiting equation is $3x^2 - q = 0$;

$$\therefore \alpha = \sqrt{\frac{q}{3}}, \quad \beta = -\sqrt{\frac{q}{3}},$$

but $f(x) = x(x^2 - q) + r$, and for both substitutions

$$x^2 - q = -\frac{2q}{3};$$

$$\therefore f(\alpha) = \sqrt{\frac{q}{3}}\left(-\frac{2q}{3}\right) + r = -2\left(\frac{q}{3}\right)^{\frac{3}{2}} + r,$$

$$f(\beta) = -\sqrt{\frac{q}{3}}\left(-\frac{2q}{3}\right) + r = 2\left(\frac{q}{3}\right)^{\frac{3}{2}} + r.$$

If then $\left(\frac{r}{2}\right)^2 < \left(\frac{q}{3}\right)^3$, $f(\alpha)$ is negative, therefore there is one root $> \alpha$; also $f(\beta)$ is positive, therefore there is one root between α and β, and another less than β; if $\left(\frac{r}{2}\right)^2 > \left(\frac{q}{3}\right)^3$, $f(\alpha)$ is positive, therefore there is no root greater than α, nor one between α and β, because $f(\beta)$ is positive; but there is one root $< \beta$, that is, one negative root which is the only real root. If $\sqrt{q}$ be written for x, the result is $+ r$; hence when all the roots are real, the greatest lies between $\sqrt{q}$ and $\sqrt{\frac{q}{3}}$.

These results were obtained by a different method (p. 40).

Ex. 2. $x^n - qx + r = 0$.

$\therefore f'(x) = nx^{n-1} - q = 0$ which has one real root α and $n-2$ imaginary ones, or two real roots α and β, and $n-3$ imaginary ones, according as n is even or odd. In the former case $f(\alpha) = \left(\frac{q}{n}\right)^{\frac{1}{n-1}}\left(\frac{q}{n} - q\right) + r = -\frac{n-1}{n}\, q \left(\frac{q}{n}\right)^{\frac{1}{n-1}} + r$ which is negative or positive according as $\left(\frac{q}{n}\right)^n >$ or $<$ $\left(\frac{r}{n-1}\right)^{n-1}$; therefore the proposed equation, which has necessarily $n-2$ imaginary roots, will have two real roots or none, according as $\left(\frac{q}{n}\right)^n >$ or $< \left(\frac{r}{n-1}\right)^{n-1}$.

In the latter case $f(\alpha) = -\frac{n-1}{n}\, q \left(\frac{q}{n}\right)^{\frac{1}{n-1}} + r$, $f(\beta) = \frac{n-1}{n}\, q \left(\frac{q}{n}\right)^{\frac{1}{n-1}} + r$, which have different or the same signs according as $\left(\frac{q}{n}\right)^n >$ or $< \left(\frac{r}{n-1}\right)^{n-1}$; therefore the proposed equation (which has necessarily $n-3$ imaginary roots) will have three real roots, or one, according as

$$\left(\frac{q}{n}\right)^n > \text{ or } < \left(\frac{r}{n-1}\right)^{n-1}.$$

55. No equation, complete or incomplete, can have more positive roots than it has changes of signs from + to — and from — to +, and no complete equation can have more negative roots than it has continuations of the same sign.

When an equation is complete, since to each term reckoning from the second corresponds either a change or a continuation of signs, the sum of the numbers expressing the changes and continuations is exactly equal to the degree of the equation.

Suppose that we have already formed the product of the factors corresponding to the imaginary and negative roots of an equation; then we shall obtain the first member of the equation by multiplying this product by the factors $x - a$, $x - b, \ldots$ corresponding to the positive roots; and if we can shew that if any polynomial, whatever be the signs of its terms, be multiplied by $x - a$, the resulting polynomial will present at least one more change of signs than the original, the proposition will be established as far as regards the positive roots. Let the signs of the polynomial be

$$+ \; - \; + \; + \; - \; - \; - \; - \; + \; - \; +$$

and let it be multiplied by a factor $x - a$; then, writing down only the signs of the operation, we have

$$\begin{array}{lcccccccccccc} & + & - & + & + & - & - & - & - & + & - & + & \\ & & - & + & - & - & + & + & + & + & - & + & - \\ \hline \text{and} & + & - & + & \pm & - & \mp & \mp & \mp & + & - & + & - \end{array}$$

for the signs of the result, the doubtful sign $\pm$ being written where the addition of unlike signs in the partial products is to be performed. Upon comparing this result with the original, and upon considering that, of the partial products, the first has the same signs as the original polynomial, and the second the contrary and advanced one row towards the right, we see that here, and in every other case, the continuations are

replaced by ambiguities, the other signs remaining the same as in the original polynomial, with the exception of the final sign, which is superadded and is always contrary to the last sign of the original polynomial. Hençe in the case of a single ambiguity (as for example the 4th term in the above instance,) taking the upper sign, the series of signs will be unaltered; and taking the lower, although a change will be lost in passing to the next term, yet one has already been introduced; and in the case of a group of ambiguities, (the 6th, 7th, and 8th terms in the above instance) although, taking the lower sign, a change will be lost at the end of it, one at least must previously have been gained, but none lost, in the preceding part of it; so that however the ambiguities, whether occurring singly or in groups, are taken, no change can be lost. As, therefore, in passing from the multiplicand to the product, it is the continuations only of the former which are altered, and as it is impossible that the number of changes can be diminished, however it may be increased; therefore in the most unfavourable case the number of changes will remain the same as before, and in this case, if the original polynomial terminate with a change, the superadded sign will introduce another change in the product; but if it terminate with a continuation, then the corresponding ambiguity will form a change either with the preceding or superadded sign. Every other positive root, in like manner, will introduce at least one change of sign. It follows therefore that no equation, complete or incomplete, can have a greater number of positive roots than it has changes of signs.

To prove the second part of the proposition, change x into $-y$; then if the equation be complete, the continuations will be replaced by changes, and *vice versâ*; and by the preceding proof the transformed equation cannot have more positive roots than it has changes; and therefore the proposed cannot have a greater number of negative roots than it has continuations. This is

Des Cartes's rule of signs, and is applicable, as we see, to discover a limit to the number of positive roots of every equation; but not to discover a limit to the number of negative roots, unless the equation be complete, or unless we supply the deficient powers of x, each of which we may consider as having ± 0 for its coefficient. But for the negative roots, the best practical way is to write $-y$ for x, and to find the limit to the number of positive roots of the transformed equation.

56. When a complete equation has all its roots real, the number of changes is exactly equal to the number of positive roots, and the number of continuations to the number of negative roots. For if m, r, be respectively the number of positive and negative roots, and m', r', the number of changes and continuations, $m + r = m' + r'$, each of these being equal to the degree of the equation; and as m cannot exceed m', nor r exceed r', the only way in which this equation can exist is $m = m'$, $r = r'$.

57. In incomplete equations the above theorem will often enable us to detect the presence of imaginary roots.

Ex. 1. $x^3 + qx + r = 0$. This equation has visibly (supposing q and r essentially positive) no positive root, and one negative root (Art. 10); if we complete it, it becomes $x^3 \pm 0x^2 + qx + r = 0$, and taking the lower sign there is only one continuation of signs, and consequently only one negative root, which is therefore the only real root of the equation.

Ex. 2. $x^5 - 2x^2 + 1 = 0$. A limit of the number of positive roots is 2; and writing $-y$ for x, we get $y^5 + 2y^2 - 1 = 0$ a limit of the number of positive roots of which is 1, or the number of negative values of x cannot exceed 1; therefore the equation has at least two imaginary roots.

58. Every equation, which, otherwise complete, wants a term between two terms of the same sign, has at least two impossible roots; and every equation, which, otherwise complete, wants t terms between two terms of the same sign, has at least $t+1$ or t impossible roots, according as t is odd or even.

Let the equation be

$$x^n + p_1 x^{n-1} + \ldots + Px^{r+t+1} + Qx^r + \ldots + p_{n-1}x + p_n = 0,$$

where P and Q have the same sign; then writing that sign before all the intermediate evanescent terms, let s = number of changes and $n - s$ = number of continuations presented by the equation, which are limits respectively of the number of positive and negative roots. Now make the signs of all the intermediate evanescent terms alternately positive and negative, so that $t + 1$ or t fresh changes may be introduced according as t is odd or even; then $n - s - t - 1$ and $n - s - t$ are limits of the number of negative roots. Hence there cannot be more than s positive roots, and $n - s - t - 1$ or $n - s - t$ negative roots, or more than $n - t - 1$ or $n - t$ possible roots; and therefore there are at least $t + 1$ or t impossible roots according as t is odd or even. Similarly, if t terms are wanting between two terms of different signs, it may be shewn that there are at least $t - 1$ or t impossible roots, according as t is odd or even. If $t = 1$, or if only one term be wanting between two terms of the same sign, then the equation has at least two impossible roots; but if a term be wanting between two terms of contrary signs, we cannot conclude any thing respecting the nature of its roots.

PROB. Shew that the equation

$$(x - a)(x - b)(x - c) - a'^2(x - a) - b'^2(x - b) - c'^2(x - c) = 2a'b'c',$$

has for a limiting equation the quadratic to which it is reduced by making any two of the quanties a', b', c', vanish; and thence that all its roots are real.

SECTION IV.

ON THE DEPRESSION OF EQUATIONS SOME OF WHOSE ROOTS HAVE PARTICULAR RELATIONS TO EACH OTHER, OR ARE OF A PECULIAR FORM.

Equal Roots.

59. Among the cases in which an equation may be depressed by reason of particular relations existing among its roots, the most important is that where the polynomial which forms its first member has equal factors, or where the equation has equal roots; because, both in the methods of determining the number and situation of the real roots of an equation, and also of approximating to the values of its incommensurable roots, one condition either essential or advantageous is, that the roots should be all different from one another, or that the equation should contain no equal roots. We must therefore shew how we may be assured that a proposed equation has no equal roots; and when it has equal roots, we must shew how they may be found, and, consequently, the complete solution of the equation made to depend upon that of one or several equations having only unequal roots.

60. If the polynomial $f(x)$ and its derived function of the first order $f'(x)$ have no common measure, the equation

$f(x) = 0$ has no equal roots; but if they have a common measure, the equation has equal roots, every simple factor of the common measure occurring one more times in $f(x)$ than it does in the common measure.

Let $a, b, c, \ldots l$ be all the roots real or imaginary of $f(x) = 0$, then, changing x into $y + h$, we have

$$f(y+h) = (y+h-a)(y+h-b)\ldots(y+h-l);$$

now if each member be expanded and arranged according to powers of y, the coefficient of y in the first member is $f'(h)$ (Art. 26), and in the second member it is

$$(h-b)(h-c)\ldots(h-l)+(h-a)(h-c)\ldots(h-l)$$
$$+(h-a)(h-b)\ldots(h-l)+\ldots$$

each of the factors $h-a$, $h-b$, ... being left out in succession; therefore, equating these coefficients and replacing h by x, we have

$$f'(x) = (x-b)(x-c)\ldots(x-l)+(x-a)(x-c)\ldots(x-l)$$
$$+(x-a)(x-b)\ldots(x-l)+\ldots$$

Hence if $f(x)$ has only one factor $= x-a$, $f'(x)$ is not divisible by $x-a$, because one of its terms does not involve $x-a$; and in the same manner it may be proved that any other of the unequal factors of $f(x)$ is not a divisor of $f'(x)$. Therefore if $f(x)$ be composed of unequal factors, $f(x)$ and $f'(x)$ have no common measure.

Again, $f'(x) = (x-a)(x-b)(x-c)\ldots(x-l)\times$

$$\left\{\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\ldots+\frac{1}{x-l}\right\}.$$

Now suppose the equation $f(x) = 0$ to have m roots equal to a, r roots equal to b, p roots equal to c,

$$\therefore f'(x) = (x-a)^m(x-b)^r(x-c)^p\ldots(x-l)\times$$

$$\left\{\frac{m}{x-a}+\frac{r}{x-b}+\frac{p}{x-c}+\ldots+\frac{1}{x-l}\right\}.$$

Therefore $f'(x)$ is divisible by $(x-a)^{m-1}(x-b)^{r-1}(x-c)^{p-1}$; and therefore, if $f(x)$ has equal factors, $f(x)$ and $f'(x)$ have a common measure, formed by the product of all those factors, each raised to a power less by unity than that to which it is raised in $f(x)$.

61. Hence if we know the value of one of the equal roots of an equation, we may find its multiplicity, that is, the number of times it is repeated, by substituting it in the derived functions taken in order; then the degree of the first of the derived functions which does not vanish by the substitution, expresses its multiplicity.

For suppose the factor $x - a$ to be repeated m times,
$\therefore f(x) = (x-a)^m \cdot \phi(x)$, where $\phi(x)$ has no factor $= x - a$.

Change x into $a + h$, then (Art. 27),

$$h^m.\phi(a+h) = f(a) + f'(a)\frac{h}{1} + f''(a)\frac{h^2}{1.2} + \ldots + f^m(a)\frac{h^m}{\lfloor m} + \ldots + h^n.$$

Now the first member is divisible by h^m, but by no higher power, therefore the second member is so, and therefore $f'(a) = 0$, $f''(a) = 0, \ldots f^{m-1}(a) = 0$; but $f^m(a)$ is a finite quantity, because the coefficient of h^m is so in the first member, that is, the first of the derived function which does not vanish for $x = a$, is that whose order is m, the number of times the root is repeated.

62. To decompose a polynomial having equal factors, into other polynomials which have only unequal factors.

Let $f(x) = X_1 X_2^2 X_3^3 \ldots . X_m^m$

where X_1 denotes the product of the factors which enter only once, X_2 the product of those which enter twice, ..., and X_m the product of those which enter m times; then if $f_1(x)$ denote the greatest common measure of $f(x)$ and $f'(x)$

$$f_1(x) = X_2 X_3^2 X_4^3 \ldots . X_m^{m-1}.$$

Again, treating the polynomial $f_1(x)$ in the same manner as $f(x)$ was treated, if $f_2(x)$ denote the greatest common measure of $f_1(x)$ and $f_1'(x)$

$$f_2(x) = X_3 X_4^2 \ldots \ldots X_m^{m-2},$$

and proceeding in this manner we shall at last come to

$$f_{m-1}(x) = X_m,$$

beyond which, if the process be continued, we find $f_m(x) = 1$, as X_m has only unequal factors. Hence, by division, we obtain

$$\frac{f(x)}{f_1(x)} = X_1 X_2 X_3 \ldots X_m = \phi_1(x) \text{ suppose,}$$

$$\frac{f_1(x)}{f_2(x)} = X_2 X_3 \ldots \ldots X_m = \phi_2(x),$$

$$\frac{f_2(x)}{f_3(x)} = X_3 X_4 \ldots \ldots X_m = \phi_3(x),$$

$$\ldots = \ldots\ldots\ldots\ldots = \ldots$$

$$\frac{f_{m-2}(x)}{f_{m-1}(x)} = X_{m-1} X_m = \phi_{m-1}(x),$$

$$\frac{f_{m-1}(x)}{f_m(x)} = X_m = \phi_m(x).$$

Hence $\dfrac{\phi_1(x)}{\phi_2(x)} = X_1, \dfrac{\phi_2(x)}{\phi_3(x)} = X_2, \ldots = \ldots$

$$\frac{\phi_{m-1}(x)}{\phi_m(x)} = X_{m-1}, \quad \phi_m(x) = X_m.$$

The solution of the original equation is thus reduced to that of the equations $X_1 = 0$, $X_2 = 0$, $\ldots\ldots X_m = 0$, each of which contains only unequal roots.

63. Hence the process of decomposing a polynomial $f(x)$ that has equal factors may be thus represented,

$$\begin{array}{llll} f(x) & f_1(x) \ldots\ldots f_{m-1}(x) & & f_m(x) \\ \phi_1(x) & \phi_2(x) \ldots\ldots \phi_m(x) & & \\ X_1 & X_2 \ldots\ldots X_m. & & \end{array}$$

In the first line each term, beginning with $f_1(x)$, is the greatest common measure of the preceding term and its derived function, and the last term $f_m(x)$ is unity; in the second line each term is the quotient of the division of that term of the first line under which it stands by the following term; and in the third line each term is the quotient of the division of that term of the second line under which it stands by the following term, and any term may equal unity. Then each of the functions X_1, X_2, ... X_m will, by its subscribed index, shew the multiplicity of the factors of which it is composed in the original polynomial; and, by its degree, the number of factors that have that multiplicity; and if any one of them X_r equals unity, then $f(x)$ admits no factor occurring r times.

Ex. 1.

$$f(x) = x^8-7x^7-2x^6+118x^5-259x^4-83x^3+612x^2 -108x-432,$$
$$f_1(x) = x^4-7x^3+13x^2+3x-18,$$
$$f_2(x) = x-3,$$
$$f_3(x) = 1,$$
$$\phi_1(x) = x^4-15x^2+10x+24,$$
$$\phi_2(x) = x^3-4x^2+x+6,$$
$$\phi_3(x) = x-3,$$
$$X_1 = x+4,$$
$$X_2 = x^2-x-2,$$
$$X_3 = x-3,$$
$$\therefore f(x) = (x+4)(x^2-x-2)^2(x-3)^3.$$

Ex. 2. $x^n - qx + r = 0$ will have a pair of equal roots if $\left(\frac{q}{n}\right)^n = \left(\frac{r}{n-1}\right)^{n-1}$.

The limiting equation is $nx^{n-1} - q = 0$;

$\therefore x = \left(\frac{q}{n}\right)^{\frac{1}{n-1}}$ is the value of the equal roots, if the equation admits any; substituting it in the proposed, $x(x^{n-1} - q) + r = 0$, we find

$$\left(\frac{q}{n}\right)^{\frac{1}{n-1}} \left(\frac{q}{n} - q\right) + r = 0 \dots (1),$$

$$\text{or } \left(\frac{q}{n}\right)^{n} = \left(\frac{r}{n-1}\right)^{n-1}$$

for the relation among the coefficients, in order that the proposed may admit a pair of equal roots. Moreover when n is even, r must be positive, and the root which recurs is $\left(\frac{q}{n}\right)^{\frac{1}{n-1}}$; when n is odd, r may be either positive or negative, but in the former case the root which recurs will be $+\left(\frac{q}{n}\right)^{\frac{1}{n-1}}$, and in the latter $-\left(\frac{q}{n}\right)^{\frac{1}{n-1}}$ as appears from (1).

Commensurable Roots.

64. Commensurable roots are those whose exact values can be expressed by finite numbers either whole or fractional, and therefore of course not involving any irrational quantity. When the coefficients are whole numbers, and that of the first term unity, the commensurable roots are necessarily whole numbers, as will be proved; in other cases they may be fractions; but in all cases they can be readily obtained, and the equation depressed.

65. If the coefficients of any equation be whole numbers, the equation can have only whole numbers for its commensurable roots.

If possible, let $\frac{a}{b}$, a fraction in its lowest terms, be a commensurable root of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n = 0,$$

$$\text{then } \left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \ldots\ldots + p_n = 0;$$

therefore, multiplying by b^{n-1} and transposing,

$$-\frac{a^n}{b} = p_1 a^{n-1} + p_2 a^{n-2} b + \ldots\ldots + p_n b^{n-1},$$

that is, a fraction in its lowest terms is equal to a whole number, which is impossible; therefore $\frac{a}{b}$ is not a root of the equation.

If therefore the equation can be satisfied by real quantities, since they are not expressible in the form of a vulgar fraction, they must be either whole numbers or interminable decimals. Hence the commensurable roots can only be whole numbers; and the other real roots are incommensurable; that is, they cannot be expressed by finite rational numbers, either whole or fractional, and therefore can never be exactly known; but their values may be approximated to with any degree of accuracy, as will be shewn.

66. The commensurable roots of $f(x) = 0$, which are necessarily whole numbers, may be always found by the following process, called the Method of Divisors, proposed by *Newton*.

Suppose a to be an integral root; then, substituting a for x, and reversing the order of the terms, we have

$$p_n + p_{n-1} a + p_{n-2} a^2 + \ldots\ldots + p_1 a^{n-1} + a^n = 0;$$

$$\therefore \frac{p_n}{a} + p_{n-1} + p_{n-2} a \ + \ldots\ldots + p_1 a^{n-2} + a^{n-1} = 0.$$

Hence $\frac{p_n}{a}$ is an integer which we may denote by q_1; substituting, and dividing again by a,

$$\frac{q_1 + p_{n-1}}{a} + p_{n-2} + \ldots + p_1 a^{n-3} + a^{n-2} = 0.$$

Similarly, $\frac{q_1 + p_{n-1}}{a}$ is an integer $= q_2$ suppose; and proceeding in this manner, we shall at last arrive at

$$\frac{q_{n-1} + p_1}{a} + 1 = 0.$$

Hence that a may be a root of the equation, the last term p_n must be divisible by it, so must the sum of the quotient and next coefficient, $q_1 + p_{n-1}$; and continuing the uniform operation, the sum of each coefficient and the preceding quotient must be divisible by a, the final result being always -1.

If therefore we take the quotients of the division of the last term by each of the divisors of the last term which are comprised within the limits of the roots, and add these quotients to the coefficient of the last term but one; divide these sums by the respective divisors, add the new quotients which are integers (neglecting the others) to the next coefficient and divide by the respective divisors; and so on through all the coefficients (dropping every divisor as soon as it gives a fractional quotient), those divisors of the last term which give -1 for a final result are the integral roots of the equation; and we shall thus obtain all the integral roots, unless the equation have equal roots, the test of which will be that some of the roots already found satisfy $f'(x) = 0$; and the number of times that any one is repeated will be expressed by the degree of the first of the derived functions which that root does not reduce to zero, when written in it for x. It is best to ascertain by direct substitution whether $+1$ and -1 are roots, and so to exclude them from the divisors to be tried.

Ex. $$x^3 + 3x^2 - 8x + 10 = 0.$$

Here the roots lie between $\frac{8}{4} + 1$ and -11 (Arts. 44, 42), and the divisors of the last term are $\pm \{2, 5, 10\}$,

$\therefore a =$	2	-2	-5	-10
$q_1 =$	5	-5	-2	-1
$q_1 + (-8) =$	-3	-13	-10	-9
$q_2 =$	$\times$	$\times$	2	$\times$
$q_2 + 3 =$			5	
$q_3 =$			$-1.$	

Therefore -5 is the only commensurable root, since it does not satisfy the equation $f'(x) = 3x^2 + 6x - 8 = 0$.

67. The number of divisors to be tried may be diminished by observing, that if the roots of $f(x) = 0$ were diminished by any whole number m, the last term of the transformed equation $f(y + m) = 0$ would be $f(m)$; if therefore a were an integral value of x, $a - m$ would be an integral value of y, and would be therefore a divisor of $f(m)$. Hence any divisor, a, of the last term of $f(x)$ is to be rejected which does not satisfy the condition $\frac{f(m)}{a - m} =$ an integer, when for m any integer, such as ± 1, ± 10, is substituted.

Ex. $$x^3 - 5x^2 - 18x + 72 = 0.$$

Changing the signs of the alternate terms, we have

$$x^3 + 5x^2 - 18x - 72 = 0 \quad \text{or } x^3 - 72 + 5x\left(x - \frac{18}{5}\right) = 0,$$

therefore the roots lie between 19 and -5.

But $$f(1) = 50, \quad f(-1) = 84,$$

and the only admissible divisors of 72 which, when diminished

by 1, divide 50 are

$$6,\ 3,\ 2,\ -4,$$

also the only admissible divisors increased by 1 which divide 84 are

$$6,\ 3,\ -2,\ -3,$$

$$\therefore\ 6,\ 3,\ 2,\ -2,\ -3,\ -4$$

are the only divisors which need to be tried.

68. If a proposed equation have fractional coefficients, or if its first term be affected with a coefficient, since (Art. 30) it can be transformed into another equation with first term unity and every coefficient a whole number, this method will enable us to find the commensurable roots of every equation under a rational form. If the coefficients be whole numbers and the first term be p_0x^n, and we only wish to find the roots which are integers, no transformation will be necessary; only every divisor of the last term which is a root, will lead to a result $-p_0$ instead of -1.

Ex. $6x^4 - 25x^3 + 26x^2 + 4x - 8 = 0.$

It is the same as

$$(x-2)^2\,(3x-2)\,(2x+1) = 0.$$

Solution of Reciprocal Equations.

69. These are equations which are not altered by changing x into $\frac{1}{x}$, and of which the roots are consequently of the form $a,\ \frac{1}{a},\ b,\ \frac{1}{b},\ \ldots$ together with $+1$ or -1 several times repeated. The particular form of the equation necessary to satisfy this condition, investigated at Art. 33, is such as to permit a great simplification in its solution; when the degree does not exceed the ninth, the solution can be completely effected.

In (34) it is proved that every reciprocal equation of an odd order will have $x + 1$ or $x - 1$ for a factor, according as its last term is positive or negative; and that every reciprocal equation of an even order with its last term negative (and consequently having no middle term) will have $x^2 - 1$ for a factor; and that if these factors be expelled, the depressed equation, in both cases, will be a reciprocal equation of an even order with its last term positive; this therefore we shall assume as the standard form of reciprocal equations.

70. The roots of a reciprocal equation of an even number of dimensions exceeding a quadratic, may be found by the solution of an equation of half the number of dimensions.

Let the equation be

$$x^{2n} + px^{2n-1} + qx^{2n-2} + \dots + kx^{n+1} + lx^n + kx^{n-1} + \dots$$
$$\dots + qx^2 + px + 1 = 0,$$

then collecting the terms which are equidistant from the extremities in pairs, and dividing by x^n, we have

$$x^n + \frac{1}{x^n} + p\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + \dots + k\left(x + \frac{1}{x}\right) + l = 0.$$

Let $x + \dfrac{1}{x} = y$, then because

$$x^{m+1} + \frac{1}{x^{m+1}} = y\left(x^m + \frac{1}{x^m}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right),$$

making $m = 1, 2, 3 \dots n$, successively, and substituting in each equation from the preceding, we have

$$x^2 + \frac{1}{x^2} = y^2 - 2$$

$$x^3 + \frac{1}{x^3} = y(y^2-2) - y = y^3 - 3y$$

$$x^4 + \frac{1}{x^4} = y\,(y^3 - 3y) - (y^2 - 2) = y^4 - 4y^2 + 2$$

$$\ldots = \ldots\ldots\ldots = \ldots$$

$$x^n + \frac{1}{x^n} = y^n - ny^{n-2} + \ldots$$

Hence, by substitution, the original equation will be transformed into an equation of n dimensions in y; any root of which, a, will give two roots of the original equation, by means of the relation $x + \frac{1}{x} = a$, and a quadratic factor, $x^2 - ax + 1$.

Ex. 1.

$$x^9 + x^8 - 9x^7 + 3x^6 - 8x^5 - 8x^4 + 3x^3 - 9x^2 + x + 1 = 0.$$

Expelling the root -1, by means of Art. 6, we have for the depressed equation

$$x^8 + (-1+1)\,x^7 + (0-9)\,x^6 + (9+3)\,x^5 + (-12-8)\,x^4$$
$$+ (20-8)\,x^3 + (-12+3)x^2 + (9-9)x + 1 = 0,$$

or $x^4 + \frac{1}{x^4} - 9\left(x^2 + \frac{1}{x^2}\right) + 12\left(x + \frac{1}{x}\right) - 20 = 0,$

or $y^4 - 4y^2 + 2 - 9\,(y^2 - 2) + 12y - 20 = 0,$

or $y^4 - 13y^2 + 12y = 0;$

$\therefore\ y = 0,\ y = 1$, and the other roots are $3,\ -4$;

therefore the proposed equation is

$$(x+1)(x^2+1)(x^2-x+1)(x^2-3x+1)(x^2+4x+1) = 0.$$

Ex. 2. $\quad 2x^6 - 5x^5 + \overset{4}{\cancel{8}}x^4 - \overset{4}{\cancel{8}}x^2 + 5x - 2 = 0.$

Expelling the factor $x^2 - 1$, the depressed equation is

$$2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0,$$

which may be resolved into $(x - 1)^2\,(2x^2 - x + 2) = 0.$

71. It may be observed, that by precisely the same process, the equation

$$x^{2n} + px^{2n-1} + \dots . kx^{n+1} + lx^n + kmx^{n-1} + hm^2x^{n-2} + \dots$$
$$\dots + pm^{n-1}x + m^n = 0,$$

admits of the same reduction as the recurring equation which it becomes when $m = 1$; the formulæ to be used being

$$x + \frac{m}{x} = y, \; x^{n+1} + \left(\frac{m}{x}\right)^{n+1} = y\left\{x^n + \left(\frac{m}{x}\right)^n\right\} - m\left\{x^{n-1} + \left(\frac{m}{x}\right)^{n-1}\right\}.$$

Solution of Binomial Equations.

72. These are equations of the form $x^n \pm a = 0$, containing only a single power of the unknown quantity, which may be reduced to reciprocal equations; for let α be the arithmetical value of $\sqrt[n]{a}$, and for x write αx, then the equation becomes $x^n \pm 1 = 0$, which is reciprocal.

Although we have already obtained the complete solution of this equation by Trigonometry, (p. 16), so that with the aid of tables of sines, the numerical values of the roots may be easily found in the form $a + b\sqrt{-1}$ as approximately as can be desired, yet the solution by a purely algebraical process deserves attention, since in it additional properties of the roots are brought to light; and these roots, that is, the n^{th} roots of unity or of negative unity, are not unfrequently employed in several of the higher branches of analysis.

73. In all cases of the equation $x^n \pm 1 = 0$, having expelled the real factors if there be any, if we transform it by the substitution $y = x + \frac{1}{x}$, so that $x^2 - yx + 1 = 0$, since y will be the sum of a pair of conjugate roots, it will always be real, as every value of x is impossible; and therefore the equation will be transformed into another of half the number of dimensions having all its roots real.

74. If a be an imaginary root of $x^n - 1 = 0$, then a^m will be a root, m being any number positive or negative.

For since a is a root, $a^n = 1$; therefore $(a^n)^m = 1$ or $(a^m)^n - 1 = 0$; therefore a^m is a root.

Also, if a be an imaginary root of $x^n + 1 = 0$, then a^m is also a root, m being any *odd* number positive or negative.

$$\text{For } a^n = -1, \ \therefore (a^n)^m = (-1)^m = -1, \text{ since } m \text{ is odd,}$$
$$\text{or } (a^m)^n + 1 = 0, \ \therefore a^m \text{ is a root.}$$

In both cases, all the roots are manifestly unequal, for the derived function nx^{n-1} can have no factor in common with $x^n \pm 1$.

75. The equations $x^n - 1 = 0$ and $x^m - 1 = 0$ can have no other common root, except unity, m and n being prime to each other.

For suppose, if possible, a to be another common root, and let a and b be two numbers determined so as to satisfy the equation $an - bm = 1$, which can always be done, since m and n are prime to one another; then $a^n = 1$, $a^m = 1$, $a^{an} = 1$, $a^{bm} = 1$, $\therefore a^{an-bm} = 1$, or $a = 1$, which is consequently the only common root. This would also appear if we sought the greatest common measure of $x^n - 1$ and $x^m - 1$, as we should find only $x - 1$.

76. The imaginary roots of $x^n - 1 = 0$, n being a prime number, are the same as the several powers of a from 1 to $n - 1$, a being any one of the imaginary roots.

For the quantities $a, a^2, a^3, \ldots a^{n-1}$ are roots by what has been proved, and no two of them are equal; for if possible let $a^p = a^q$, p and q being both less than n, therefore $a^{p-q} = 1$; or a is a root of $x^{p-q} - 1 = 0$, and also of $x^n - 1 = 0$, which is impossible, because $p - q$ and n are prime to one another;

therefore the roots of the equation are all exhibited in the series

$$1,\ a,\ a^2,\ \ldots\ .\ a^{n-1};$$

and if it be continued, the roots recur in the same order, for

$$a^n = 1,\ a^{n+1} = a^n \cdot a = a,\ a^{n+2} = a^n \cdot a^2 = a^2 \ldots .$$

77. This property of producing all the other roots by its different powers, which, when n is prime, belongs to all the imaginary roots, is in other cases generally confined to the first imaginary root, a, determined by *De Moivre's* formula, (as proved, p. 19,) or its conjugate; or rather to any root a^m, provided m be prime to n, or to its conjugate. If therefore β be any root, it is always true that any power of β is a root; but not always true that all the roots can be represented by powers of β.

Thus in the case $x^6 - 1 = 0$ or $(x^3 - 1)(x^3 + 1) = 0$, if we take

$$\beta = \frac{-1 + \sqrt{-3}}{2},$$

we can, by its powers from 0 to 5, only produce the roots of $x^3 - 1 = 0$ twice over; but if we take

$$a = \frac{1 + \sqrt{-3}}{2} = \cos\frac{2\pi}{6} + \sqrt{-1}\sin\frac{2\pi}{6},$$

we can, by the powers of a, produce all the six roots.

78. The solution of $x^n - 1 = 0$, where n is a composite number, may be always reduced to those of $x^p - 1 = 0$, $x^q - 1 = 0, \ldots .$ where $p, q, r, \ldots .$ are all the prime factors of n.

First, suppose $n = pq$, then $x^n - 1$ or $x^{pq} - 1$ is divisible, both by $x^p - 1$ and $x^q - 1$; therefore the roots of $x^p - 1 = 0$,

$$1,\ a,\ a^2,\ \ldots\ .\ a^{p-1},$$

and the roots of $x^q - 1 = 0$,

$$1, \beta, \beta^2, \ldots. \beta^{q-1}$$

are roots of $x^n - 1 = 0$; also the products, formed by multiplying every quantity in the first series by every quantity in the second, are roots; for each one of these will be of the form $a^r\beta^s$, also $a^{rn} = 1$, $\beta^{sn} = 1$; $\therefore (a^r\beta^s)^n = 1$, consequently $a^r\beta^s$ is a root; and no two of these products are alike, for, if possible, suppose $a^r\beta^s = a^\rho\beta^\sigma$; $\therefore a^{r-\rho} = \beta^{\sigma-s}$, but $a^{r-\rho}$ is a root of $x^p - 1 = 0$ and $\beta^{\sigma-s}$ of $x^q - 1 = 0$, therefore these equations have another root in common besides unity, which is impossible; therefore the pq products formed by multiplying every root of $x^p - 1 = 0$ by every root of $x^q - 1 = 0$ are the roots of the equation $x^{pq} - 1 = 0$.

In the same manner if $n = pqr$, and a, β, γ be respectively roots of $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$, it may be shewn that $a^\pi\beta^\sigma\gamma^\tau$ is the general form of the roots of $x^n - 1 = 0$, and will give all the roots, if π, σ, τ assume all values from 0 to $p-1$, $q-1$, $r-1$, respectively.

79. Secondly, suppose $n = p^2$, and let the roots of $x^p - 1 = 0$ be

$$1, a, a^2, \ldots. a^{p-1},$$

then these, as well as

$$1, \sqrt[p]{a}, \sqrt[p]{a^2}, \ldots. \sqrt[p]{a^{p-1}},$$

are roots of the proposed, as is also the product of every one of the first row by every one of the second; therefore we have p^2 quantities of the general form $a^r\sqrt[p]{a^s}$, all satisfying the proposed and all different from one another, which are therefore the roots of the equation.

Similarly if $n = p^3$, the general expression for the roots would be $a^r \sqrt[p]{a^s}, \sqrt[p^2]{a^t}$, a being a root of $x^p - 1 = 0$. And if $n = p^2qr$, and a, β, γ denote respectively roots of the

equations $x^p - 1 = 0$, $x^q - 1 = 0$, $x^r - 1 = 0$, then the general form of the root of the proposed will be $a^{\pi}.\sqrt[p]{a^{\rho}}.\beta^{\sigma}.\gamma^{\tau}$, and all the n roots will result by giving to π and ρ all values from 0 to $p - 1$, and to σ and τ all values from 0 to $q - 1$ and $r - 1$, respectively. And, in the most general case, where $n = p^{\mu} q^{\nu} r^{\pi} \ldots$, the roots might in the same manner be found by combining the roots of the equations $x^p - 1 = 0$, $x^q - 1 = 0, \ldots$

80. Thus the solution of $x^n - 1 = 0$ can always be reduced to the case where n is a prime number; and the case of n a prime number, by a method invented by *Gauss*, may be made to depend upon the solution of equations whose degrees do not exceed the greatest prime number which is a divisor of $n - 1$. The leading feature of *Gauss's* method is to represent the imaginary roots by a series of powers of any one of them, whose indices form a geometrical instead of an arithmetical progression. Thus, if m be a number (and such can always be found) whose several powers from 1 to $n - 1$, when divided by n, have different remainders, and a be any imaginary root, then all the roots may manifestly be represented by

$$a^m,\ a^{m^2},\ a^{m^3},\ \ldots\ a^{m^{n-1}}.$$

81. Any radical has always as many values as there are units in its index, and these values are obtained by multiplying the arithmetical value of the root of the quantity under the sign, by each of the roots of $+ 1$ or $- 1$.

For every root of the equation $x^n \pm a = 0$ is an algebraical value of $\sqrt[n]{\mp a}$; but, whatever be $\pm a$, this equation admits n roots all different from one another, therefore the radical $\sqrt[n]{\pm a}$, considered algebraically, will have n different

values. When a is real and positive, the equation $x^n = a$ α has always one real root $\underline{a}$, and the n values of $\sqrt[n]{a}$ will be obtained by multiplying $\underline{a}$ by each of the n values of $\sqrt[n]{1}$; in like manner the values of $\sqrt[n]{-a}$ will result from multiplying $\underline{a}$ by the values of $\sqrt[n]{-1}$.

82. Hence $\sqrt[m]{a} \times \sqrt[n]{b}$ will have r values, where r is the least common multiple of m and n.

For let α, β, be the arithmetical values of the radicals,

$$\text{then } \sqrt[m]{a} \times \sqrt[n]{b} = \alpha\beta\,(1)^{\frac{m+n}{mn}};$$

but if $\dfrac{m+n}{mn}$ be reduced to its lowest terms, the numerator will be an integer and the denominator will be r, the least common multiple of m and n;

$$\therefore\ \sqrt[m]{a} \times \sqrt[n]{b} = \alpha\beta\,(1)^{\frac{1}{r}}$$

which has r different values.

SECTION V.

ON THE GENERAL SOLUTION OF EQUATIONS OF A DEGREE INFERIOR TO THE FIFTH.

83. We shall now direct our attention to certain cases, in which a solution has been effected, of finding the expressions for all the roots of an equation of an assigned degree in terms of its coefficients, the coefficients being general symbols. These methods which, as was before observed, succeed only for equations of which the degree does not exceed the fourth, are the results of particular artifices; but they are all reducible to one principle, as will be shewn.

Solution of a quadratic.

84. Let the equation be reduced to the form

$$x^2 + px + q = 0,$$

then this may be transformed into $y^2 = a$, by taking away its second term. For putting $x = y - \frac{1}{2}p$, (Art. 29,) we have

$$y^2 - py + \frac{p^2}{4} + py - \frac{p^2}{2} + q = 0,$$

or $y^2 = \frac{p^2}{4} - q$;

$$\therefore x = -\frac{p}{2} + y = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

If $\frac{p^2}{4} > q$, the roots are real; if $\frac{p^2}{4} = q$ they are equal, and $x^2 + px + q = (x + \frac{1}{2}p)^2$ is a perfect square;
if $\frac{p^2}{4} < q$, the roots are impossible.

Hence any trinomial $ax^2 + bx + c$ or $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ will be resolvable into two real simple factors or not, according as $\frac{b^2}{4a^2} >$ or $< \frac{c}{a}$; and will be a perfect square when $\frac{b^2}{4a^2} = \frac{c}{a}$, or $b^2 = 4ac$, *i.e.* when the square of the coefficient of the middle term is equal to four times the product of the coefficients of the extreme terms.

85. Any impossible expression of the form $\alpha \pm \beta\sqrt{-1}$ may be transformed into $r(\cos\theta \pm \sqrt{-1}\sin\theta)$.

For, α and β being real quantities, there always exists an angle θ, such that $\tan\theta = \frac{\beta}{\alpha}$;

$$\text{then } \cos\theta = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \ \sin\theta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}};$$

if therefore $\sqrt{\alpha^2 + \beta^2} = r$, which is called the modulus of the quantity $\alpha + \beta\sqrt{-1}$, we have

$$\alpha \pm \beta\sqrt{-1} = r(\cos\theta \pm \sqrt{-1}\sin\theta).$$

Hence any pair of imaginary roots of an equation may be represented by the formula $r(\cos\theta \pm \sqrt{-1}\sin\theta)$.

In the case of the expression $-\frac{p}{2} \pm \sqrt{-1}\sqrt{q - \frac{p^2}{4}}$,

$$r = \sqrt{q}, \ \cos\theta = -\frac{p}{2\sqrt{q}};$$

$$\text{hence } x^2 + px + q = x^2 - 2r\cos\theta x + r^2,$$

that is, any irreducible quadratic factor of an equation,

$$x^2 + px + q, \ \text{ where } \frac{p^2}{4} < q,$$

may be transformed into $x^2 - 2r\cos\theta x + r^2$, where $r = \sqrt{q}$ and $\cos\theta = -\dfrac{p}{2\sqrt{q}}$.

86. To solve an equation of the form

$$x^{2n} + px^n + q = 0.$$

Putting $x^n = y$, we find $y^2 + py + q = 0$.

If this have two real roots a and b, then the $2n$ values of x are the roots of the equations

$$y^n - a = 0, \ y^n - b = 0.$$

If the roots of the quadratic are imaginary, *i.e.* if $\dfrac{p^2}{4} < q$, then, making $r = \sqrt{q}$, and $\cos\theta = \dfrac{-p}{2\sqrt{q}}$, the proposed equation becomes

$$x^{2n} - 2r\cos\theta x^n + r^2 = 0,$$

$$\text{or } x^{2n} - 2\cos\theta x^n + 1 = 0,$$

changing x^n into $x^n r$, which has already been solved, p. 24.

Solution of a cubic equation by Cardan's rule.

87. Let the equation be reduced to the form

$$x^3 + qx + r = 0,$$

and put $x = y + z$, that is, suppose x equal to the sum of two other unknown quantities,

$$\therefore\ x^3 = 3yz\,(y + z) + y^3 + z^3,$$

and therefore the proposed equation becomes

$$(3yz + q)\,(y + z) + y^3 + z^3 + r = 0.$$

Now since we have two unknown quantities, and have made only one supposition respecting them, namely, that $y + z = x$, we are at liberty to make another; let therefore $3yz + q = 0$,

$$\text{or } y^3z^3 = -\left(\frac{q}{3}\right)^3,\quad \therefore\ y^3 + z^3 = -r.$$

Hence y^3, z^3, are roots of the equation

$$t^2 + rt - \left(\frac{q}{3}\right)^3 = 0,$$

since the second term with its sign changed is equal to their sum, and the last term is equal to their product. Solving this equation, we have

$$t = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}};$$

$$\therefore\ y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}},\quad z^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$$

and $x = y + z =$

$$\left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}},$$

an expression which (since the cube root of any quantity has three values) contains implicitly the three roots.

88. This solution only extends to those cases in which the cubic has two impossible roots.

Let m and n be the arithmetical values of the two surds in the value of x, and 1, a, a^2, the three cube roots of unity; then the three values of y are (Art. 81) m, am, a^2m, and

those of z, n, an, a^2n. By combining these values two and two to form $y + z$, we shall have nine values of x, the number being tripled by reason of our having employed $y^3z^3 = -\left(\frac{q}{3}\right)^3$, instead of $yz = -\frac{q}{3}$, the relation arising immediately in the process; and we observe that every combination will satisfy $y^3z^3 = -\left(\frac{q}{3}\right)^3$, but only three the given condition $yz = -\frac{q}{3}$, which latter are the roots, viz.

$$m + n,\ am + a^2n,\ a^2m + an;$$

or, substituting for a, a^2, their values $-\frac{1}{2}(1 \pm \sqrt{-3})$ (p. 15),

$$m + n \text{ and } -\tfrac{1}{2}\{m + n \pm (m - n)\sqrt{-3}\}.$$

Hence, as long as the expression $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ is possible, the values of m and n are possible, and the equation has one possible root, the numerical value of which, as also those of the two imaginary roots, may be obtained from the above formulæ; but when the expression $\sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ is impossible, m and n are impossible, and all the three roots appear under imaginary forms; whereas the equation, being of an odd degree, has at least one real root, and indeed, since $\frac{r^2}{4} + \frac{q^3}{27}$ is negative, has (Art. 54) all its roots real; in this case therefore the above formulæ, although algebraical expressions for the roots, cannot, on account of the imaginary quantities which they involve, be applied to furnish the numerical values of the roots.

89. In the case of the roots being all real, which for the reason just stated is called the irreducible case, that is, when

q is negative and $\frac{q^3}{27} > \frac{r^2}{4}$, it may be observed that the assumptions in the process

$$y^3 + z^3 = -r, \quad y^3 z^3 = +\left(\frac{q}{3}\right)^3,$$

are inconsistent with one another; for the product of two real quantities can never exceed the square of half their sum. In this case we can shew that in the expressions for the roots, the impossible quantities destroy one another, and the three roots are real. For let the values of m^3 and n^3 be represented by $a \pm b\sqrt{-1}$, then expanding by the binomial theorem, and taking P and Q to denote real functions of a and b, we have

$$(a \pm b\sqrt{-1})^{\frac{1}{3}} = \mathrm{P} \pm \mathrm{Q}\sqrt{-1};$$

$$\therefore m + n = 2\mathrm{P}, \quad m - n = 2\mathrm{Q}\sqrt{-1},$$

and the three values of x are $2\mathrm{P}$ and $-\frac{1}{2}(2\mathrm{P} \pm 2\mathrm{Q}\sqrt{3})$ which are all real. This mode of proceeding however is useless in finding the numerical values of the roots; for if we convert $(a + b\sqrt{-1})^{\frac{1}{3}}$ into a series, P and Q will be expressed by series which rarely converge and from which we can never obtain the exact values of P and Q; and if we attempt to express the cube root of $a \pm b\sqrt{-1}$ by an expression of the same form, we shall have to solve a cubic of the same kind as the one in question.

90. Hence *Cardan's* rule succeeds for the following forms, where q and r are essentially positive,

$$x^3 + qx \pm r = 0 \text{ in all cases,}$$

$$x^3 - qx \pm r = 0 \text{ when } \frac{q^3}{27} < \frac{r^2}{4},$$

and fails for $x^3 - qx \pm r = 0$ when $\frac{q^3}{27} > \frac{r^2}{4}$, all the roots of which are real.

91. There is one case in which *Cardan's* rule succeeds for the equation $x^3 - qx + r = 0$ when all the roots are real. It is when the roots of the reducing quadratic are equal; for then $m = n$ and the values of x are $m + n$, $-\frac{1}{2}(m + n)$, $-\frac{1}{2}(m+n)$. In this case $\frac{r^2}{4} = \frac{q^3}{27}$ or $\frac{r}{2} = \left(\frac{q}{3}\right)^{\frac{3}{2}}$, $\therefore m^3 = n^3 = -\frac{r}{2} = -\left(\frac{q}{3}\right)^{\frac{3}{2}}$, $\therefore m + n = -2\sqrt{\frac{q}{3}}$, and the roots are $-2\sqrt{\frac{q}{3}}, \sqrt{\frac{q}{3}}, \sqrt{\frac{q}{3}}$.

92. If in the expression

$$\left(-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}}$$

$$= \sqrt{\frac{q}{3}}\left\{-\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}} \pm \sqrt{\frac{r^2}{4}\left(\frac{3}{q}\right)^3 + 1}\right\}^{\frac{1}{3}},$$

we put $\cot\phi = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, it becomes $\sqrt{\frac{q}{3}}\,(-\cot\phi \pm \operatorname{cosec}\phi)^{\frac{1}{3}}$.

Hence, reducing, the real root of $x^3 + qx + r = 0$ is

$$\sqrt{\frac{q}{3}}\left(\tan^{\frac{1}{3}}\frac{\phi}{2} - \cot^{\frac{1}{3}}\frac{\phi}{2}\right),$$

which may be further transformed into

$$-2\sqrt{\frac{q}{3}}\cot 2\theta, \text{ by putting } \tan\frac{\phi}{2} = \tan^3\theta.$$

Similarly, the real root of $x^3 - qx + r = 0$, $\frac{q^3}{27} < \frac{r^2}{4}$,

becomes (by putting $\operatorname{cosec}\phi = \frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, $\tan\frac{\phi}{2} = \tan^3\theta$),

$$-2\sqrt{\frac{q}{3}}\operatorname{cosec} 2\theta.$$

Also in the irreducible case, $x^3 - qx \pm r = 0$, $\frac{q^3}{27} > \frac{r^2}{4}$, the expression

$$\mp\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}} = \left(\frac{q}{3}\right)^{\frac{3}{2}}\left\{\mp\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}} \pm \sqrt{-1}\left(1 - \frac{27r^2}{4q^3}\right)^{\frac{1}{2}}\right\},$$

making $\cos\phi = \pm\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$, becomes

$$\left(\frac{q}{3}\right)^{\frac{3}{2}}\{-\cos\phi \pm \sqrt{-1}\sin\phi\}$$

$$\text{or } \left(\frac{q}{3}\right)^{\frac{3}{2}}\{\cos(\pi \pm \phi) \pm \sqrt{-1}\sin(\pi \pm \phi)\};$$

therefore the three values of x are

$$-2\sqrt{\frac{q}{3}}\cos\frac{\phi}{3}, \quad 2\sqrt{\frac{q}{3}}\cos\left(\frac{\pi \pm \phi}{3}\right).$$

Solution of a Biquadratic Equation by Des Cartes's method.

93. Let the proposed equation be reduced to the form

$$x^4 + qx^2 + rx + s = 0,$$

and as the first member may be always regarded as the product of two real quadratic factors, assume it

$$= (x^2 + px + f)(x^2 - px + g)$$
$$= x^4 + (g + f - p^2)x^2 + (pg - pf)x + fg$$

(effecting the multiplication,) where the coefficients of the second terms p and $-p$ are equal and of opposite signs, because the second term of the proposed equation is wanting, that is, the sum of its roots is zero. Hence equating coefficients

$$g+f-p^2=q,\quad pg-pf=r,\quad fg=s,$$

$$\text{or } g+f=q+p^2,\quad g-f=\frac{r}{p};$$

$$\therefore\ 2g=q+p^2+\frac{r}{p},\quad 2f=q+p^2-\frac{r}{p};$$

$$\therefore\ 4fg=q^2+2qp^2+p^4-\frac{r^2}{p^2}=4s,$$

$$\text{or } p^6+2qp^4+(q^2-4s)\,p^2-r^2=0,$$

the equation for determining p, which rises to the sixth degree, because a polynomial of four dimensions, may (Art. 17) be resolved $\frac{4.3}{1.2}$, or six ways into two quadratic factors. Also, because the values of p are the sums of every two roots of the proposed biquadratic, and because the sum of these roots is zero, and therefore the sum of any two is equal and of a contrary sign to the sum of the other two, therefore the values of p will be in pairs differing only in sign; this is the reason why the equation for determining p involves only even powers of p, and may therefore be depressed to a cubic by putting $p^2=y$. The reducing cubic is

$$y^3+2qy^2+(q^2-4s)\,y-r^2=0,$$

which (Art. 10) has necessarily one real positive root; let this be e^2, then the four values of x are contained in the quadratic equations

$$x^2+ex+\frac{1}{2}\left(q+e^2-\frac{r}{e}\right)=0$$

$$x^2-ex+\frac{1}{2}\left(q+e^2+\frac{r}{e}\right)=0.$$

94. The reducing cubic will have all its roots real, unless two of the roots of the proposed biquadratic are possible, and two impossible.

For the square of the sum of any two roots of the proposed is a root of the reducing cubic; if therefore the proposed have all its roots real, the reducing cubic will have all its roots real; ~~or if the proposed have all its roots imaginary, then the sum of each pair of conjugate roots will be real, and therefore the cubic will have two real roots, and consequently all its roots real.~~ But if the proposed have two real and two imaginary roots, then the sum of a real and an imaginary root will be imaginary, and therefore the cubic will have one and consequently two imaginary roots. As it is only in the latter case that a solution of the reducing cubic can be obtained, therefore *Des Cartes's* method can only be applied to those cases in which two roots of the biquadratic are possible and two impossible.

95. If the roots of the reducing cubic can be obtained, and are put under the forms $(2\alpha)^2$, $(2\beta)^2$, $(2\gamma)^2$, then the four roots of the biquadratic are

$$-(\alpha+\beta+\gamma),\ \beta+\gamma-\alpha,\ \alpha+\gamma-\beta,\ \alpha+\beta-\gamma.$$

For $-\frac{1}{2}q = \alpha^2+\beta^2+\gamma^2$ and $r^2 = (8\alpha\beta\gamma)^2$,

$$\text{let } p^2 = (2\alpha)^2 \text{ or } p = \pm 2\alpha;$$

therefore, taking the upper sign,

$$f = \frac{1}{2}\left(q+p^2-\frac{r}{p}\right) = -(\alpha^2+\beta^2+\gamma^2)+2\alpha^2-2\beta\gamma$$
$$= \alpha^2-(\beta+\gamma)^2;$$

therefore the first reducing quadratic is

$$x^2+2\alpha x+\alpha^2-(\beta+\gamma)^2 = 0,$$

which gives for x the values

$$-(\alpha+\beta+\gamma),\ \beta+\gamma-\alpha;$$

similarly the other quadratic, taking $p=-2\alpha$, is

$$x^2-2\alpha x+\alpha^2-(\beta-\gamma)^2=0,$$

which gives the other values

$$\alpha+\gamma-\beta,\ \alpha+\beta-\gamma.$$

Hence the roots of the biquadratic are symmetrical functions of the roots of the reducing cubic; and whatever root of the reducing cubic is used in the process, the same values of x are obtained.

Solution of a complete Biquadratic.

96. Let the equation be

$$x^4+px^3+qx^2+rx+s=0,$$

and let it be supposed the same as

$$\left(x^2+\frac{p}{2}x+m\right)^2-(kx+l)^2=0,$$

where k, l, m are unknown, and are to be determined so as to make the latter equation coincide with the proposed. Now

$$\begin{aligned}\left(x^2+\frac{p}{2}x+m\right)^2 &= x^4+px^3+\left(\frac{p^2}{4}+2m\right)x^2+pmx+m^2\\ -(kx+l)^2 &= \qquad\qquad -k^2x^2-2klx-l^2;\end{aligned}$$

therefore by comparing this with the proposed we have, to determine k, l, m, the equations

$$\frac{p^2}{4}+2m-k^2=q,\quad pm-2kl=r,\quad m^2-l^2=s.$$

Substituting in the second the values of k and l obtained from the first and third, we have

$$8m^3 - 4qm^2 + (2pr - 8s)\,m - p^2 s + 4qs - r^2 = 0,$$

which will necessarily give one real value for m; then k and l are known, and we find the two reducing quadratics

$$x^2 + \left(\frac{p}{2} + k\right)x + m + l = 0$$

$$x^2 + \left(\frac{p}{2} - k\right)x + m - l = 0.$$

97. This method can be employed only when two roots of the biquadratic are possible and two impossible; for suppose the roots to be α, β, γ, δ, and suppose any two α, β, to satisfy the first reducing quadratic, and consequently γ, δ, the second,

$$\therefore\ m + l = \alpha\beta,\quad m - l = \gamma\delta;$$

$\therefore\ m = \frac{1}{2}(\alpha\beta + \gamma\delta)$, and the other values of m must be $\frac{1}{2}(\alpha\gamma + \beta\delta)$, $\frac{1}{2}(\alpha\delta + \beta\gamma)$.

Hence if α, β, γ, δ, be either all possible or all impossible, the values of m are real; but if two roots of the biquadratic be possible and two impossible, then two values of m will be impossible, and the reducing cubic may be solved by *Cardan's* rule.

Solution of a Biquadratic by Euler's method.

98. Let the equation be reduced to the form

$$x^4 + qx^2 + rx + s = 0,$$

and assume $x = y + z + u$;

$$\therefore\ x^2 = y^2 + z^2 + u^2 + 2\,(yz + yu + zu),$$

$$\text{or } x^2 - (y^2 + z^2 + u^2) = 2\,(yz + yu + zu);$$

$$\therefore\ x^4 - 2x^2(y^2 + z^2 + u^2) + (y^2 + z^2 + u^2)^2 = 4\,(y^2z^2 + y^2u^2 + z^2u^2) \\ + 8yzu\,(y + z + u),$$

or, replacing $y + z + u$ by x and transposing,

$$x^4 - 2x^2(y^2+z^2+u^2) - 8yzux + (y^2+z^2+u^2)^2$$
$$-4(y^2z^2+y^2u^2+z^2u^2) = 0.$$

In order that this may coincide with the proposed we must have

$$q = -2(y^2+z^2+u^2), \quad r = -8yzu, \quad s = (y^2+z^2+u^2)^2$$
$$-4(y^2z^2+y^2u^2+z^2u^2),$$

$$\text{or } y^2+z^2+u^2 = -\frac{q}{2}, \; y^2z^2+y^2u^2+z^2u^2 = \frac{q^2-4s}{16},$$

$$yzu = -\frac{r}{8} \text{ or } y^2z^2u^2 = \frac{r^2}{64};$$

hence y^2, z^2, u^2 are roots of the cubic

$$t^3 + \frac{q}{2}t^2 + \frac{q^2-4s}{16}t - \frac{r^2}{64} = 0.$$

Let t^2, t'^2, t''^2 denote the three values of t in this equation;

$$\therefore y = \pm t, \quad z = \pm t', \quad u = \pm t'',$$

which six values combined three and three would give 8 values of $y + z + u$ or x, instead of 4, the number being doubled because we have used $y^2z^2u^2 = \frac{r^2}{64}$, instead of the given condition $yzu = -\frac{r}{8}$ which only allows those values of y, z, u to be combined which give, when multiplied together, a result with a contrary sign to r.

Hence if r be negative there must be two or no negative quantities in every combination, and if r be positive there must be two or no positive quantities, in every combination representing a root. Therefore, in the former case, that is, when r is negative, the roots are

$$t - t' - t'', \quad t' - t - t'', \quad t'' - t - t', \quad t + t' + t'',$$

and in the latter case, when r is positive, the roots are

$$t+t'-t'',\quad t+t''-t',\quad t'+t''-t,\quad -t-t'-t'',$$

and it will be observed, that the second set of roots results from the first by changing the sign of any one of the quantities t, t', t''.

99. In this case also the reducing cubic will have all its roots real, except when the proposed has two possible and two impossible roots.

Since the last term of the reducing cubic is essentially negative, it will always have one real positive root t^2, and the remaining roots will be either both positive, both negative, or impossible; that is, of the forms

$$t'^2,\ t''^2;\ -t'^2,\ -t''^2;\ \text{or } \rho^2(\cos 2\theta \pm \sqrt{-1}\sin 2\theta).$$

Hence, according as the reducing cubic has three positive roots, two negative roots, or impossible roots, the biquadratic

$$x^4 + qx^2 - rx + s = 0$$

will have its roots respectively of the forms

$$t \pm (t' + t''),\quad -t \pm (t' - t'')$$
$$t \pm \sqrt{-1}\,(t'-t''),\quad -t\pm\sqrt{-1}\,(t'+t'')$$
$$t \pm 2\rho\cos\theta,\quad -t\pm\sqrt{-1}\,2\rho\sin\theta.$$

In the case of the equation

$$x^4 + qx^2 + rx + s = 0$$

we must change the sign of t in the above expressions, and the results will be its roots.

SECTION VI.

ON THE SEPARATION OF THE ROOTS OF EQUATIONS.

100. The propositions in the preceding sections lead us to several important conclusions relating to the nature and the limits of the roots of every equation; and for equations of low degrees and of certain particular forms, the methods detailed in them (especially that of Art. 49) will actually determine the number and situation of all the real roots, that is, two quantities between which each of the real roots lies. They still, however, leave unsolved the main problem, which is to discover the number and situation of the real roots of an equation of any degree. This we shall now endeavour to effect by the methods proposed by *M. M. Sturm* and *Fourier*, which are among the greatest improvements recently made in the Theory of Equations.

Sturm's method of separating the Roots.

101. By performing a process nearly the same as that of finding the greatest common measure of $f(x)$, and its first derived function $f'(x)$, a series of expressions may be obtained, in which, by simply substituting a and b successively for x, the number of roots of $f(x) = 0$ which lie between a and b may be exactly determined. The enunciation and proof are as follows.

Let $f(x) = 0$ be an equation of n dimensions cleared of equal roots, $f_1(x)$ the first derived function of $f(x)$; and let the process of finding the greatest common measure of $f(x)$ and $f_1(x)$ be performed with the condition that the remainder after each operation has its sign changed, and so modified is used for the divisor of the next operation; (this changing of the signs of the remainders, which would be indifferent if the object was only to discover the greatest common measure of $f(x)$ and $f_1(x)$, is essential to the method we are about to explain); and let $f_2(x)$, $f_3(x)$, ... $f_n(x)$ be the series of modified remainders; then the difference of the number of changes of sign, in the results of the substitutions of a and b for x in the series of quantities

$$f(x),\ f_1(x),\ f_2(x),\ \ldots\ f_n(x),\ (1),$$

expresses the number of real roots of $f(x) = 0$, which lie between a and b.

Calling the successive quotients q_1, q_2, ... we shall have the equations

$$\begin{aligned} f(x) &= q_1 f_1(x) - f_2(x) \\ f_1(x) &= q_2 f_2(x) - f_3(x) \\ \ldots &= \ldots\ldots \\ f_{m-1}(x) &= q_m f_m(x) - f_{m+1}(x) \\ \ldots &= \ldots\ldots \\ f_{n-2}(x) &= q_{n-1} f_{n-1}(x) - f_n(x), \end{aligned}$$

$f_n(x)$ being necessarily a number (Art. 60), since $f(x) = 0$ has no equal roots; which shew, first, that no value of x can make two consecutive functions $f_{m-1}(x)$ and $f_m(x)$ vanish, for then $f_{m+1}(x)$ and all the succeeding functions would vanish, which is impossible, since the last is a number; and, secondly, that any value which makes a function $f_m(x)$ vanish, reduces the two adjacent ones to the same numerical value with different signs. Now if in series (1) we make $x = c$, and then suppose

c to assume all possible ascending values from $-\infty$ to $+\infty$, the resulting series of signs will have two states of permanence; one, as long as c is nearer to $-\infty$, and the other after c is nearer to $+\infty$, than any quantity which makes any one of the expressions in series (1) vanish; and between these states, whenever any of the expressions vanish, alterations in the order or number of changes of signs, or in both, will occur; and we shall shew that when x passes through a quantity which makes one or more of the auxiliary functions vanish, it is only the order but not the number of changes which is affected; and that when x passes through a root of $f(x) = 0$, then a change of sign is lost.

First, let $x = c$ make only one of the auxiliary functions $f_m(x)$ vanish, without making $f(x)$ vanish; then to discover the effect, upon the series of signs, of passing through c, we must compare the results of substituting $c - h$ and $c + h$ for x, h being as small as ever we please; therefore we may suppose h so small that neither $f(x)$ nor any of the auxiliary functions can vanish for values between $c - h$ and $c + h$, and that the sign of any series ascending by powers of h depends upon that of its first term. Hence the only part of series (1) in which the passage from $c - h$ to $c + h$ can produce any effect upon the series of signs, is

$$f_{m-1}(x),\ f_m(x),\ f_{m+1}(x),$$

in which, if we write $c-h$ for x, expand the results (Art. 27), and reserve only that term of each on which its sign depends, we have

$$\mp f_{m-1}(c),\ \ -hf'_m(c), \pm f_{m+1}(c),$$

which, since the extremes have different signs, give a change and continuation whatever be the sign of the middle term; and these, by changing the sign of h, will be replaced by a continuation and change; *i.e.* the passage from $c-h$ to $c+h$, through a root of $f_m(x) = 0$, causes an alteration in the order

but not in the number of changes. If the same value of x made an auxiliary function vanish in another part of the series, since adjacent terms can never vanish, the same considerations would shew that no change of sign could be lost or gained.

Secondly, let $x = c$ be a root of $f(x) = 0$; the substitution of $c-h$ for x in $f(x)$ and $f_1(x)$ (taking h so small that the sign of the whole of each series depends upon that of its first term, and writing down only the first terms) gives

$$-hf'(c), f_1(c), \text{ or } -hf'(c),\ f'(c),$$

which have different signs; but the same signs, if the sign of h be changed; therefore the two functions $f(x)$, $f_1(x)$, which for $x = c - h$ give a change, for $x = c + h$ give a continuation; and therefore, in passing through a root of $f(x) = 0$, a change of signs is lost. If at the same time that $f(x)$ becomes zero, any number of auxiliary functions vanished, since no two of them could be adjacent, it would follow, as before, that no change of sign could be lost in the parts of the series where they are situated.

Since then a change of signs is lost every time the substituted quantity passes through a root of $f(x) = 0$; and since a change cannot be lost in any other way, nor one ever introduced; it follows, that the excess of the number of changes given by $x = a$, above that given by $x = b$ $(a < b)$, is exactly equal to the number of real roots of $f(x) = 0$ lying between a and b.

It may be observed that if by either of these substitutions one of the auxiliary functions $f_m(x)$ is reduced to zero, it may be neglected in estimating the number of changes; for in that case, as has been shewn, the adjacent functions will have different signs, and therefore the evanescent function, with whatever sign it is taken, will cause the three to furnish but one change, and may therefore be omitted without affecting the number of changes.

102. Hence if we substitute $-\infty$ and $+\infty$ for x, or, which comes to the same thing, if we form the first terms of $f(x)$, $f_1(x)$, ... $f_n(x)$ into a series, and then change x into $-x$, the difference of the number of changes of sign in the two resulting series will express the whole number of real roots.

103. Since, in finding the greatest common measure, each remainder is usually one dimension lower than the preceding, the auxiliary functions will usually be n in number, the same as the degree of the equation, and of the several dimensions from $n-1$ to 0. When none of the auxiliary functions are wanting and the first terms of $f(x)$, $f_1(x)$, $f_2(x)$, ... $f_n(x)$ have all the same sign, $-\infty$ gives n changes and $+\infty$ gives no changes, therefore all the roots are real.

104. On the contrary, when none of the auxiliary functions are wanting and the first terms have not all the same sign, there will be as many pairs of imaginary roots as there are changes in the signs of the first terms. In the series formed by the first terms of the $n+1$ quantities $f(x)$, $f_1(x)$, ... $f_n(x)$, let there be s changes and therefore $n-s$ continuations, then these are the same as the numbers of changes and continuations produced by the substitution of $+\infty$ for x; now write $-\infty$ for x in the same series, then every change will be replaced by a continuation, and *vice versâ;* and therefore there will be $n-s$ changes, a number necessarily greater than s; that is, in passing from $-\infty$ to $+\infty$, $n-2s$ changes are lost; therefore the equation has only $n-2s$ real roots, and therefore $2s$ imaginary roots; or as many pairs of imaginary roots as there are changes of sign in the series formed by the first terms of the $n+1$ quantities $f(x)$, $f_1(x)$, ... $f_n(x)$.

105. If one of the auxiliary functions $f_m(x)$ be such as to preserve the same sign for all values of x between a and b,

then in ascertaining the number of roots between a and b, we may neglect all the auxiliary functions after $f_m(x)$. Because (since in general the passage through a quantity which makes one of the auxiliary functions vanish, causes an alteration only in the order but not in the number of changes, and since $f_m(x)$ preserves the same sign for all values of x between a and b,) the number of changes presented by the series of auxiliary functions which follow $f_m(x)$ cannot be altered by the substitution of any value of x between those limits; and therefore the difference in the number of changes given by the substitutions of a and b will be the same, whether we take the auxiliary functions that follow $f_m(x)$ into account or not.

Hence if $f_m(x) = 0$ have all its roots impossible, since $f_m(x)$ will preserve the same sign for all values of x, we may arrest the process at it, and confine our attention to the $m + 1$ functions

$$f(x),\ f_1(x),\ f_2(x),\ \ldots\ f_m(x);$$

and, as in the former case, if the first terms of these offer s changes of sign, there will be only $m - 2s$ real roots, and the rest will be imaginary.

106. We shall now give some applications of this theorem.

Having formed the auxiliary functions

$$f_1(x), f_2(x), f_3(x),\ \ldots\ f_n(x),$$

then if none of them be wanting, and their leading terms be all positive, (for the leading term of $f(x)$ is necessarily so) the equation will have all its roots real; but if the leading terms are not all positive, the equation will have as many pairs of imaginary roots as there are chaṅges of sign in them. But if some of the auxiliary functions are wanting, the number of real roots must be determined by substituting $-\infty$ and $+\infty$ for x in their leading terms, and taking the difference between the numbers of changes resulting from

these substitutions. This determines the *number* of real roots. To determine their *situations* we must substitute 0, 1, 2, 3, ... for x in the series

$$f(x),\ f_1(x),\ f_2(x)\ \ldots\ f_n(x),$$

till we arrive at a number which gives the same number of changes as is given by $+\infty$; then, by noting the difference in the number of changes produced by the extreme substitutions, we determine the number of $+$ roots; and by noting those consecutive integers between which one or more changes are lost, we determine the integral limits between which the positive roots are situated, either singly or in groups; and in the latter case we must substitute fractional quantities lying between the integral limits, smaller and smaller, till the complete separation of each group of roots is effected.

In like manner for the negative roots, we must substitute $0,\ -1,\ -2,\ -3, \ldots\ldots$ till we arrive at a number which gives the same number of changes as is given by $-\infty$; then the total number of negative roots, and an interval in which each is situated, may be determined, exactly in the same manner as for the positive roots. And in order to diminish the labour of the process, it must be observed that when, in forming the auxiliary functions, we come to one (that of the second degree, for instance, when the conditions of Art. 84 are fulfilled) which is incapable of changing its sign for any value of x, we may take it for the last of the auxiliary functions.

Ex. 1. $f(x) = x^3 - 7x + 7 = 0.$

$$f_1(x) = 3x^2 - 7)\,3x^3 - 21x + 21\,(x$$
$$\underline{3x^3 - 7x}$$
$$-14x + 21,$$

$$\text{or } f_2(x) = 2x-3)6x^2-14\,(3x+\frac{9}{2}$$
$$6x^2-9x$$
$$\overline{}$$
$$9x-14$$
$$9x-\frac{27}{2}$$
$$\overline{}$$
$$-\tfrac{1}{2},$$

or $f_3(x) = +1$.

Since the leading terms are all positive, and none of the functions are wanting, the roots are all possible. Also, since 2 makes all the functions positive, the substitutions for the purpose of separating the roots may begin from thence, therefore making $x = 2,\ 1,\ 0,\ -1,\ -2, \ldots$ the signs are as follows:

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$
(2)	+	+	+	+
(1)	+	−	−	+
(0)	+	−	−	+
(−1)	+	−	−	+
(−2)	+	+	−	+
(−3)	+	+	−	+
(−4)	−	+	−	+.

We may stop here because the signs are the same as those given by $-\infty$. Since the first line gives no changes and the second line two, two roots lie between 2 and 1; also the last line has one more change than the preceding, therefore one root lies between -3 and -4.

To separate the two roots which lie between 1 and 2, let $x = \frac{3}{2}$, then

$$\left(\frac{3}{2}\right) = \quad - \quad - \quad 0 \quad +$$

which has one more change than the first line, and one less than the second, (whatever sign we give to the zero); therefore one root lies between 1 and $1 \cdot 5$, and another between $1 \cdot 5$ and 2.

Ex. 2. $f(x) = x^n - qx + r = 0.$

$$f_1(x) = nx^{n-1} - q$$

$$nx^{n-1} - q \,\big)\, x^n - qx + r \,\left(\frac{x}{n}\right.$$

$$x^n - \frac{qx}{n}$$

$$\overline{\qquad - qx\left(1 - \frac{1}{n}\right) + r,}$$

or $f_2(x) = x - a$, rejecting the positive factor $q\left(1 - \frac{1}{n}\right)$, if $a = \frac{nr}{(n-1)q}$.

But the remainder, after dividing $n\left(x^{n-1} - \frac{q}{n}\right)$ by $x - a$, is (Art. 6) $n\left(a^{n-1} - \frac{q}{n}\right)$; $\therefore f_3(x) = -\left(\frac{r}{n-1}\right)^{n-1} + \left(\frac{q}{n}\right)^n$, rejecting the positive factor $n\left(\frac{n}{q}\right)^{n-1}$.

Now supposing $f_3(x)$ positive, $+\infty$ gives no change, and $-\infty$ gives two changes when n is even, and three changes when n is odd. Hence if $\left(\frac{q}{n}\right)^n > \left(\frac{r}{n-1}\right)^{n-1}$, the proposed equation has two, or three real roots, according as n is even or odd.

Similarly if $\left(\frac{q}{n}\right)^n < \left(\frac{r}{n-1}\right)^{n-1}$, $+\infty$ gives one change, and $-\infty$ one, or two changes according as n is even or odd; and therefore the equation has no real root, or one real root, according as n is even or odd. These results agree with those found at p. 54.

Ex. 3. $f(x) = 2x^4 - 13x^2 + 10x - 19 = 0,$

$$f_1(x) = 4x^3 - 13x + 5,$$

and we find

$$f_2(x) = 13x^2 - 15x + 38.$$

But the roots of $13x^2 - 15x + 38 = 0$ are imaginary, because $(15)^2 < 4.13.38$ (Art. 84); therefore it is sufficient to consider the above three functions, and since their leading terms give two changes for $x = -\infty$, and no change for $x = +\infty$, the equation has only two real roots.

107. It is manifest that in *Sturm's* method the labour of forming the auxiliary functions increases very rapidly with the degree of the equation; since however they can always be formed, the method will enable us infallibly to determine, not a limit to the number, but the absolute number of real roots in any proposed equation, and the consecutive integers between which they lie either singly or in determined groups, as also the intervals in which no real root can be situated; but when two or more roots are indicated in any interval, if they lie very near to one another, although the method leaves no doubt of the existence of the roots, it may be very difficult to subdivide the interval sufficiently to completely separate them.

Fourier's method of separating the Roots.

108. We shall now give another method of separating the roots proposed by *Fourier*, complete to the same extent as

the former, and having the recommendation that the auxiliary functions employed in it are $f(x)$ and its successive derived functions, which can be formed by inspection; so that the method can be applied nearly with the same ease to an equation of any degree; in particular, the intervals in which no real root can be situated are, by *Fourier's* method, immediately assigned. The enunciation and proof are as follows.

The number of real roots of $f(x) = 0$ which lie between two numbers a and b, cannot exceed the difference between the number of changes of sign in the results of the substitutions of a and b for x in the series formed by $f(x)$ and its derived functions: viz. $f(x), f'(x), f''(x), \ldots . f^n(x)$.

If none of the equations

$$f(x) = 0, \quad f'(x) = 0, \ldots .$$

have a root between a and b, it is manifest that the substitution of a and b, and any intermediate quantity in $f(x)$, $f'(x)$, ... will always produce exactly the same series of signs; but if any of these equations have roots between a and b, then changes in the series of signs will occur in substituting gradually ascending quantities from a to b; our object is to show that by such substitutions the number of changes of signs can never increase, and that one change will be lost every time the substituted quantity passes through a real root of $f(x) = 0$.

First, suppose that $x = c$, (c being some quantity between a and b) makes $f(x)$ vanish, without making any of the derived functions vanish; then the result of substituting $c + h$ for x in $f(x)$ and $f'(x)$ is

$$h . f'(c) \text{ and } f'(c)$$

(supposing h so small that the signs of the whole of the two series which express $f(c + h)$ and $f'(c + h)$ depend upon those of their first terms, and writing down only the first terms) which have different or the same signs according as h is $-$ or $+$;

therefore in passing from $c - h$ to $c + h$ through a root of the equation, a change of signs is lost. It is unnecessary to attend to the other terms of the series of derived functions, because h is supposed so small that not one of them vanishes by the substitution of any quantity between $c - h$ and $c + h$, and therefore each has the same sign for $c - h$ as for $c + h$.

Secondly, suppose that $x = c$ makes one of the derived functions $f^m(x)$ vanish, without making any other of the derived functions or $f(x)$ vanish; then the result of substituting $c + h$ for x in

$$f^{m-1}(x), \quad f^m(x), \quad f^{m+1}(x),$$

(these being the only terms which it is necessary to examine) is

$$f^{m-1}(c), \quad h\,.\,f^{m+1}(c), \quad f^{m+1}(c).$$

If then the extreme terms have the same sign, there will be two changes when h is negative, and two continuations when h is positive; if the extreme terms have contrary signs, there will be one change, and one only, whether h be negative or positive; therefore in passing from $c - h$ to $c + h$ through a value which makes one of the derived functions vanish, either two changes or none will be lost, but none ever gained.

Thirdly, suppose that $x = c$ makes r consecutive derived functions vanish, without making any other derived function or $f(x)$ vanish; then the result of the substitution of $c + h$ for x in the series

$$f^{m-r}(x), f^{m-r+1}(x), \ldots . f^{m-1}(x), f^m(x), f^{m+1}(x),$$

(these being the only terms necessary to be examined) is

$$f^{m-r}(c), \frac{h^r}{|\underline{r}} f^{m+1}(c), \ldots, \frac{h^2}{|\underline{2}} f^{m+1}(c), \frac{h}{1} f^{m+1}(c), f^{m+1}(c).$$

If then the extreme terms have the same sign, there will be r or $r + 1$ changes (according as r is even or odd) when h is negative, and no change when h is positive; if the extreme

terms have contrary signs, there will be r or $r+1$ changes (according as r is odd or even) when h is negative, and one change when h is positive; therefore in passing from $c-h$ to $c+h$ through a value which makes r consecutive derived functions vanish, r or $r \pm 1$ changes are lost (according as r is even or odd) but none ever gained.

If the vanishing group consisted of $f(x)$ and the first $r-1$ derived functions (which corresponds to r roots $=c$ in $f(x)=0$), the result of the substitution of $c+h$ for x in $f(x)\ f'(x), \ldots . f^{r-1}(x),\ f^r(x)$, would be

$$\frac{h^r}{\lfloor r} f^r(c), \frac{h^{r-1}}{\lfloor r-1} f^r(c), \quad \ldots . \frac{h}{1} f^r(c), f^r(c),$$

in which there are r changes when h is negative, and none when h is positive; therefore in passing through a root which occurs r times in the equation, r changes are lost.

Lastly, suppose the substitution of $x=c$ to produce all the above cases, that is, to make $f(x)$ and the first $r-1$ derived functions to vanish; and also to make an even group of p terms in one part of the series, and an odd group of q terms in another part, to vanish at the same time; then because the conclusions respecting the effect of the passage through c upon the series of signs in one part of the series of derived functions, are not at all influenced by what happens in consequence of the same passage at another distinct part of the series, the number of changes lost in passing from $c-h$ to $c+h$ will be $r+p+q \pm 1$.

If then two numbers a and b when substituted for x in the series $f(x)\ f'(x), \ldots . f^n(x)$ give different series of signs; since the alterations in the series of signs can only have arisen from some of the quantities vanishing for values of x between a and b; and since these can only be either $f(x)$ alone, or some derived function $f^m(x)$ alone, or some group of

derived functions, or $f(x)$ and the adjoining group of derived functions, or several or all of these at the same time; and since we have proved that in substituting gradually ascending values from a to b, a change is lost for every passage through a quantity which makes $f(x)$ vanish, but none ever gained; and that in all the other cases, although sometimes no changes are lost, yet that none are ever gained; we conclude that the number of real roots of $f(x) = 0$ which lie between two numbers a and b cannot exceed the excess of the number of changes arising from substituting $x = a$, above that arising from substituting $x = b$, in the series

$$f(x),\ f'(x), \ldots . f^n(x).$$

109. Hence if the limits a and b be $-\infty$ and $+\infty$, or two numbers the first of which gives only changes and the second only continuations, and if in the series formed by $f(x)$ and its derived functions

$$f(x), f'(x), f''(x), \ldots . f^n(x),$$

c be substituted for x and be then made to assume all values between these limits, the series of signs of the results will have the following properties; there will at first be n changes of sign, and at last no change, but n continuations; these changes disappear gradually as c increases, and when once lost can never be recovered; one change disappears every time c passes through a real unequal root of $f(x) = 0$; r changes disappear every time c passes through a root which occurs r times in $f(x) = 0$; either two or none of the changes disappear every time one only of the derived functions vanishes, without $f(x)$ vanishing at the same time; an even number p of changes disappears, every time an even group of p terms (not including the first $f(x)$) vanishes; and an even number $q \pm 1$ of changes disappears, every time an odd group of q terms (not including the first $f(x)$) vanishes.

Hence if $f(x) = 0$ have all its roots real, no value of x can make any of the derived functions vanish, and thereby exterminate changes of signs, without at the same time making $f(x)$ vanish; for if it could, since those changes can never be restored, and since a change must disappear for every passage through a real root, the total number of changes lost would surpass n, which is absurd. Whenever therefore changes disappear between values of x which do not include a root of $f(x) = 0$, corresponding to that occurrence there is an equal number of imaginary roots of $f(x) = 0$. Hence if $x = c$ produces a zero between two similar signs, or if it produces an even number p of consecutive zeros either between similar or contrary signs, there will be respectively two, or p, imaginary roots corresponding; or if it produces an odd number q of consecutive zeros, there will be $q \pm 1$ imaginary roots corresponding, according as they stand between similar or contrary signs; c of course not being a root of $f(x) = 0$.

Since the derivatives which follow any one $f^r(x)$ may be supposed to arise originally from it, it is manifest that the same conclusions respecting the roots of $f^r(x) = 0$ may be drawn from observing the part of the series of derivatives

$$f^r(x),\ f^{r+1}(x),\ \ldots\ f^n(x),$$

as were drawn respecting the root of $f(x) = 0$ from the whole series.

110. When $x = -\infty$, there are n changes; and when $x = 0$, the signs of the series of derived functions become the same as those of the coefficients

$$p_n,\ p_{n-1},\ \ldots\ p_1,\ 1;$$

let the number of changes in this series of coefficients $= k$, and therefore the number of continuations (supposing the equation complete) $= n - k$; also if $x = +\infty$ the signs are

all positive and the number of changes $= 0$. Hence between $x = -\infty$ and $x = 0$, the number of changes lost is $n - k$; therefore there cannot be more than $n - k$ negative roots, *i. e.* than the number of continuations in the series of coefficients; also between $x = 0$ and $x = \infty$, the number of changes lost is k, hence there cannot be more positive roots than k, *i. e.* than the number of changes in the series of coefficients; this is *Des Cartes's* rule of signs.

111. *Fourier's* theorem may also be presented under the following form. If an equation have m real roots between a and b, then the equation whose roots are those of the proposed, each diminished by a, has at least m more changes of signs than the equation whose roots are those of the proposed, each diminished by b.

The transformed equations would be

$$f(y + a) = 0, \quad f(y + b) = 0;$$

and if these were arranged according to ascending powers of y, the coefficients would be the values assumed by $f(x)$, $f'(x)$, . . . when a and b are written for x. Therefore, whatever number of changes of signs is lost in the series $f(x)$, $f'(x)$, ... in passing from a to b, the same is lost in passing from one transformed equation to the other; but the series for a has at least m more changes than that for b, therefore $f(y + a) = 0$ has at least m more changes than $f(y + b) = 0$.

112. To apply this method to find the intervals in which the roots of $f(x) = 0$ are to be sought, we must substitute successively for x in the series formed by $f(x)$ and its derivatives, the numbers

$$-\alpha, \ldots . -10, -1, 0, 1, 10, \ldots . \beta \ (1),$$

($-\alpha$ and $+\beta$ being the least negative and least positive numbers which give respectively only changes and continua-

tions) and observe the number of changes of sign in each result.

Let h and k be the numbers of changes of sign when any two consecutive terms in series (1), a and b, are respectively written for x; therefore $h - k$ is the number of real roots that may lie between a and b; if this equals zero, $f(x) = 0$ has no real root between a and b, and the interval is excluded; if $h - k = 1$, or any odd number, there is at least one real root between a and b; if $h - k = 2$, or any even number, there may be two or some even number, or none; the latter case will happen when, as explained above, some number between a and b makes two or some even number of changes vanish, without satisfying $f(x) = 0$. Similarly, we must examine all the other partial intervals; and when two or more roots are indicated as lying in any one, their nature must be determined by a succeeding proposition.

113. The two following examples are extracted from *Fourier's* work.

Ex. 1. Let

$$\begin{aligned} f(x) &= x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 \\ f'(x) &= 5x^4 - 12x^3 - 72x^2 + 190x - 46 \\ f''(x) &= 20x^3 - 36x^2 - 144x + 190 \\ f'''(x) &= 60x^2 - 72x - 144 \\ f^{iv}(x) &= 120x - 72 \\ f^{v}(x) &= 120. \end{aligned}$$

Hence we have the following series of signs resulting from the substitutions of -10, -1, 0, ... for x in the series of quantities

	f	f'	f''	f'''	f^{iv}	f^{v}
(-10)	$-$	$+$	$-$	$+$	$-$	$+$
(-1)	$+$	$+$	$-$	$+$		$+$
(0)	$-$	$-$	$+$	$-$		$+$
(1)	$-$	$+$	$+$		$+$	$+$
(10)	$+$	$+$	$+$	$+$	$+$	$+$.

Hence all the roots lie between -10 and $+10$, because five changes have disappeared; one root lies in each of the intervals -10 to -1, and -1 to 0, because in each of them a single change is lost; no root lies between 0 and 1, because no change is lost between those limits; and three roots may be sought between 1 and 10 (because three changes have disappeared) one of which is certainly real; it is doubful whether the other two are real or imaginary.

When any value of x, c, makes one of the derived functions $f^m(x)$ vanish, we may substitute $c \pm h$ instead of c, h being indefinitely small; then all the other functions will have the same sign as when $x = c$, and the sign of $f^m(c \pm h)$ will depend upon that of $\pm hf^{m+1}(c)$; *i.e.* it will be the same or contrary to that of the following derivative $f^{m+1}(c)$, according as h is positive or negative, or according as we substitute a quantity a little less or a little greater than the value which makes it vanish. The use of this remark will be seen in the following example.

Ex. 2. $f(x) = x^4 - 4x^3 - 3x + 23 = 0$

$$f'(x) = 4x^3 - 12x^2 - 3$$

$$f''(x) = 12x^2 - 24x$$

$$f'''(x) = 24x - 24$$

$$f^{IV}(x) = 24.$$

	f	f'	f''	f'''	f^{IV}
$x = 0$	$+$	$-$	0	$-$	$+$
$x = 0 \mp h$	$+$	$-$	$\pm$	$-$	$+$
$x = 1$	$+$	$-$	$-$	0	$+$
$x = 1 \mp h$	$+$	$-$	$-$	$\mp$	$+$
$x = 10$	$+$	$+$	$+$	$+$	$+.$

Every value less than 0 gives results alternately $+$ and $-$, therefore there is no real negative root; for $x = 0$ we have a result zero placed between two similar signs, and therefore corresponding to it there is a pair of imaginary roots. There is no root between 0 and 1, but there may be two roots between 1 and 10.

114. This process will determine the intervals in which the roots are to be sought but not always their nature; when an even number of roots is indicated, they may all turn out impossible. The series of magnitudes between $-\infty$ and $+\infty$ to be substituted for x in the derived functions, has been divided into intervals of two sorts each contained by assigned limits a and b. The first sort of interval is one within which no root is comprehended; *i. e.* the limits of which give the same number of changes of signs in the series of derived functions. The second sort is one within which roots may lie, *i. e.* where the number of changes resulting from the substitution of b is less than the number resulting from the substitution of a in the series of derived functions. This second sort of interval has two subdivisions, viz. cases where the indicated roots do really exist, and others where they are imaginary. When we have ascertained that a certain number of roots may lie between a and b, we may substitute c (a quantity between a and b) in the series of derived functions, and if any changes disappear, our interval is broken into two others; if no changes disappear, we may increase or diminish c, and make a second substitution, and it may still happen that no change is lost, and so on continually; and we may be left after all in a state of uncertainty whether the separation of the roots is impossible because they are imaginary, or only retarded because their difference is extremely small. Hence when we know that two limits may include a certain number of roots, we must have

a special rule for determining whether they are possible or impossible; this has been given by *Fourier* in the two following propositions; in proving which, we assume that the development of $f(x+h)$ in Art. (27) may be put under the following forms, so as to exhibit the remainder of the series when we take only one, two, &c. terms;

$$f(x+h) = f(x) + hf'(\lambda)$$
$$f(x+h) = f(x) + hf'(x) + \tfrac{1}{2}h^2 f''(\mu),$$

and so on, where $\lambda, \mu, \ldots.$ are quantities certainly situated between x and $x+h$, but of which the exact values are unknown, and for our purpose are unnecessary.

115. Having given that between two limits a and b, $f''(x) = 0$ has no root at all, $f'(x) = 0$ one root and no more, and that $f(x) = 0$ may have either two roots or none, to discover whether these roots exist or not.

Hence, by what has already been proved, the series of signs resulting from substituting a in the series of quantities

$$f(x),\quad f'(x),\quad f''(x),\ \ldots.\ f^n(x),$$

will present two more changes than the series resulting from the substitution of b; also, if we leave out the first term, there will be one more change for a than for b; and if we leave out the first two terms, there will be exactly the same number of changes for a as for b. Therefore $f(x)$ and $f''(x)$ will be both constantly positive or constantly negative for a and b, and for all intermediate values; and $f'(x)$ will have a sign different from that of $f(x)$ and $f''(x)$ when $x = a$, and the same as that of $f(x)$ and $f''(x)$ when $x = b$.

The two roots of $f(x) = 0$, indicated as lying between a and b, will be real or imaginary according as it is or is not possible to find a quantity c, between a and b, such that $f(c)$ shall have a sign contrary to that which is common to $f(a)$ and $f(b)$.

Let therefore, if possible,

$$c = a + h = b - k$$

be a quantity between a and b, such that

$$\frac{f(c)}{f(a)} = \text{neg.} \quad \frac{f(c)}{(b)} = \text{neg.},$$

or expanding so that the terms of the second order may include the remainder of each series, and denoting by λ, μ quantities intermediate to a and b,

$$1 + h\frac{f'(a)}{f(a)} + \frac{h^2}{2}\frac{f''(\lambda)}{f(a)} = \text{neg.}$$

$$1 - k\frac{f'(b)}{f(b)} + \frac{k^2}{2}\frac{f''(\mu)}{f(b)} = \text{neg.}$$

Or since, under the given conditions, the last fraction in each line must be positive, and also $\frac{f'(a)}{f(a)} = \text{neg.}, \frac{f'(b)}{f(b)} = \text{pos.}$, we must have

$$\frac{f(a)}{f'(a)} + h = \text{pos.} \quad \frac{f(b)}{f'(b)} - k = \text{neg.};$$

$$\therefore \frac{f(a)}{f'(a)} - \frac{f(b)}{f'(b)} + h + k = \text{pos.}$$

$$\text{or } h + k = b - a > \frac{f(b)}{f'(b)} - \frac{f(a)}{f'(a)}.$$

If then this condition can be satisfied, a quantity c between a and b may exist so as to make $f(c)$ of a sign contrary to $f(a)$ and $f(b)$; and if it can be found, the indicated roots are real and are separated: but if the condition is not satisfied, that is, if the difference of the limits be equal to or less than the sum of the fractions $\frac{f(a)}{f'(a)}, \frac{f(b)}{f'(b)}$, taken without regard to sign, no such value of c exists, and the indicated roots are imaginary. It is manifest that if any three consecutive deriva-

tives $f^r(x)$, $f^{r+1}(x)$, $f^{r+2}(x)$ satisfy the prescribed conditions for a given interval, the same process will determine the nature of the pair of roots of $f^r(x) = 0$ indicated in that interval.

116. When the above condition is satisfied, we must substitute a quantity c between a and b in $f(x)$, if this has a sign contrary to the common sign of $f(a)$ and $f(b)$, the separation is effected; if not, we infer that the limits are not sufficiently close to determine the nature of the indicated roots by a single process. In the latter case $f'(c)$ necessarily differs in sign from one or the other of $f'(a)$, $f'(b)$; choosing, then, that limit which makes $f'(x)$ have a contrary sign from $f'(c)$, we must with it and c repeat exactly the same process, and we are certain at last to discover either that no roots exist in the interval, or to separate them if they do.

Ex. $x^5 - 3x^4 - 24x^3 + 95x^2 - 46x - 101 = 0.$

	f	f'	f''	f'''	f^{iv}	f^{v}
(2)	$-$	$+$	$-$	$-$	$+$	$+$
	2	1	0			
(3)	$-$	$-$	$-$	$+$	$+$	$+$.

Here, since there are two more changes for $x = 2$ than for $x = 3$, one more, omitting the first term, and the same number, omitting the two first terms, the equation may have two roots between 2 and 3, and the conditions respecting the roots of $f'(x) = 0$, $f''(x) = 0$ are satisfied; and since for the two limits, the fraction

$$\frac{f(x)}{f'(x)} \text{ becomes } \frac{7}{10} \text{ and } \frac{32}{43},$$

the sum of which is greater than the difference, 1, of the limits; therefore the two indicated roots are imaginary.

117. In the next proposition it will be necessary, for any proposed interval, to know the number of roots which each derivative when formed into an equation may have in that interval. The best practical way of doing this is, in the two series of signs produced by the two limits, to write over each sign the number of changes presented by the series commencing with the sign of the last derivative and terminating with that sign; and then to take the difference between each number in the upper line and the corresponding one in the lower. Applying the process to the foregoing example, we have

	3	2	1	1	0	0
(2)	−	+	−	−	+	+
	2	1	0	1	0	0
(3)	−	−	−	+	+	+
	1	1	1	0	0	0

where the series of indices 2, 1, 0, 1, 0, 0, mark the number of roots which the equations $f(x) = 0, f'(x) = 0, f''(x) = 0$, ... may have between the limits 2 and 3. Also we observe that in this series (and indeed in every case, if we consider the way in which they are formed,) any index has immediately adjacent to it either the same, or one differing from it by the addition of $\pm$ 1.

118. When any number of roots of $f(x) = 0$ are indicated as lying between a and b, this interval may always be broken up into others, in which such of the roots as are real are situated singly.

From observing the number of changes lost in the series formed by $f(x)$ and all its derivatives, and also in the series formed by each of the derivatives and all those which follow it, in passing from a to b, let the number of roots which $f(x) = 0$

may have, or which the derivatives taken in order when formed into equations may have, between those limits, be determined; and let them be δ, δ', δ'', ... Now suppose that in the series (where each function is accompanied by its index, *i.e.* the number of roots which, when formed into an equation, it may have between a and b)

$$\begin{array}{cccccccc} f(x) & f'(x) & f''(x) & \ldots\ldots & f^{r-1}(x) & f^r(x) & f^{r+1}(x), & \ldots\ldots \\ \delta & \delta' & \delta'' & & 2 & 1 & \varepsilon & \end{array}$$

$f^r(x)$ is the first whose index is 1, then the preceding function has 2 for its index, for it cannot have 0, otherwise, since the first index is not zero, there would be some function before $f^r(x)$ having 1 for its index. Now if ε be not zero, since $f^r(x) = 0$, $f^{r+1}(x) = 0$ cannot have a common root, two new limits a', b' may be found within the former, intercepting the root of $f^r(x) = 0$, but excluding every root of $f^{r+1}(x) = 0$. Hence the interval a, b, will be broken up into the three aa', $a'b'$, $b'b$, the first and third of which give for $f^r(x)$ an index zero, and therefore an index 1 to some preceding function, and the second $a'b'$ will either make some preceding function have an index 1, or will allow $f^r(x)$ still to be the first function whose index is unity for that interval, the indices of $f^{r-1}(x)$ and $f^{r+1}(x)$ being 2 and 0.

Suppose the latter to be the case; then, by Art. 115, we may find whether $f^{r-1}(x) = 0$ has two real roots or none between a' and b'; if there are two real roots, then taking a quantity c' between them, the interval $a'b'$ is divided into the two $a'c'$ and $c'b'$, each of which makes $f^{r-1}(x)$ or some preceding function have an index 1; but if the two roots of $f^{r-1}(x) = 0$, indicated as lying between a' and b', are imaginary, since every quantity intermediate to a' and b' will make $f^{r-1}(x)$ and $f^{r+1}(x)$ have the same sign, therefore in passing from a' to b' through the root of $f^r(x) = 0$, since the adjacent

functions have the same sign, two changes will be lost. Hence we may diminish the indices of all the preceding functions by 2, and proceed, relative to the interval $a'b'$, with that function preceding $f^r(x)$ which first has 1 for its index. Hence the proposed interval is replaced by partial intervals, in each of which the separation of the included roots is more nearly effected than in the original interval; and by proceeding with the partial intervals in the same manner as we did for a, b, we shall at last find only intervals in which the index of $f(x)$ is either 0 or 1, and the separation of the roots of $f(x) = 0$ which lie between a and b will be completely effected.

Ex. $f(x) = x^4 - x^3 + 4x^2 + x - 4 = 0,$

	f	f'	f''	f'''	f^{iv}
(-10)	+	−	+	−	+
(-1)	−	−	+	−	+
(0)	−	+	+	−	+
	3	2	2	1	0
(1)	+	+	+	+	+.

There is one root between -10 and -1, and no root between -1 and 0, also three are indicated between 0 and 1; but forming the series of indices for that interval, we see that $f''(x) = 0$ is the first equation to which the criterion can be applied; also $\dfrac{f''(x)}{f'''(x)}$ becomes $\dfrac{6x^2 - 3x + 4}{12x - 3}$ which for $x = 0$ becomes $-\dfrac{4}{3}$, and this neglecting sign is greater than 1, the difference of the limits; therefore the roots are imaginary, and consequently there is only one root of the proposed equation between 0 and 1.

The criterion of the reality of two indicated roots in any interval may be readily deduced from geometrical considerations.

Let $y = f(x)$ be the equation to a parabolic curve, then the portion of it between $x = a$, $x = b$, (supposing these limits to satisfy all the prescribed conditions,) must have the shape PCQ, (Fig. 2) O being the origin, ONM the axis of x, PN, QM, the ordinates of its extremities having the same sign, C the single point where the tangent is parallel to the axis, and the curve through the extent PCQ being convex to the axis of x, because for that interval $f(x)$ and $f''(x)$ have the same sign. But if O′N′M′, a line parallel to ONM and cutting the curve in two points, be the axis of x, the curve will have the ordinates of its extremities of the same sign, and will have its tangent parallel to the axis of x at one single point, and $f(x)\,f''(x)$ will have the same sign for all points between P and Q; hence, for any thing that yet appears, this construction will represent the function $f(x)$ between $x = a$ and $x = b$, just as well as the former; but it is manifest that when $f(x) = 0$ has two roots between a and b, there will be two points of intersection with the axis of x, and the second is the true construction; and the former belongs to the case where there is no point of intersection, and the abscissæ of the points of intersection, that is, the roots of $f(x) = 0$, are imaginary. If we knew the exact value c of OR, we might substitute it in $f(x)$, and if the sign of the result was different from that of $f(a)$ and $f(b)$, then $f(c)$ would be represented by R′C, and we should be certain that there were two points of intersection; if the same, $f(c)$ would be represented by RC, and there would be no point of intersection. But if we can only find an approximate value of c, and the sign of $f(c)$ is the same as that of $f(a)$ and $f(b)$, we are uncertain whether the points of intersection are imaginary, or so near to one another that our approximate foot of the least ordinate does not fall between them.

Now in the case of real roots, that is, when O′N′M′ is the

axis of x, and there are two points of intersection, if tangents Pt', Qs' be drawn at P, Q, it is manifest, that however near to one another the roots are, and however close the limits are to the roots, N$'$M$'$ must exceed N$'t'$ + M$'s'$, or $b - a$ must exceed $\frac{f(b)}{f'(b)} - \frac{f(a)}{f'(a)}$; if therefore we find either $\frac{f(b)}{f'(b)}$, or $-\frac{f(a)}{f'(a)}$, or their sum, greater than $b - a$, we know that the roots cannot be possible, and may pronounce them impossible. But when we find the difference of the limits greater than the sum of the subtangents, we cannot conclude that the roots are possible, for this condition is satisfied not only by the axis N$'$M$'$ but also by NM, as long as the tangents Pt, Qs, do not intersect between the curve and the axis.

In the latter case we must substitute a quantity d between a and b for x, then if $f(d)$ have a different sign from $f(a)$ and $f(b)$, the two indicated roots are real, and their separation is effected; if not, $f'(d)$ will have the same sign either as $f'(a)$ or $f'(b)$; let it be the former, then no root can lie between a and d; and we must now apply the criterion of the subtangents to the new and closer interval from d to b.

SECTION VII.

ON THE METHODS OF FINDING APPROXIMATE VALUES OF THE REAL INCOMMENSURABLE ROOTS OF EQUATIONS.

119. When all the commensurable roots of an equation have been found, and all the incommensurable roots separated by the methods explained in the foregoing sections, the next step towards the solution of the equation is to find approximate values of the incommensurable roots; and to this we shall now direct our attention.

120. It will however be necessary previously to prove certain properties of the polynomial $f(x)$ which forms the first member of the equation.

Since $f(x+h) - f(x) = f'(x)\,h + f''(x)\,\frac{h^2}{2} + \,.\,.\, + h^n$,

and as long as x is finite, none of the quantities $f'(x), f''(x)\,.\,.$ being integral functions, can become infinite, therefore by taking h sufficiently small we may make the second member as small as ever we please; consequently if x increase continuously by insensible degrees between two limits a and b, $f(x)$ will also vary continuously by insensible degrees between the same limits; and will go on increasing as long as $f'(x)$ continues positive; and when $f'(x)$ is negative, it will go on diminishing.

Again, since

$$\frac{f(x+h)-f(x)}{h}=f'(x)+f''(x)\frac{h}{2}+\ldots+h^{n-1},$$

and since by diminishing h we can make $f''(x)\frac{h}{2}+\ldots+h^{n-1}$ as near zero as ever we please, or always intermediate in value to $+e$ and $-e$ where e is as near zero as ever we please, by taking h sufficiently small we shall always have

$$\frac{f(x+h)-f(x)}{h}>f'(x)-e \;<f'(x)+e;$$

if therefore x be always taken between a and b, and if A, B, denote the least and greatest values which $f'(x)$ can assume between those limits, *à fortiori*, the following inequality may be satisfied,

$$f(x+h)-f(x)>h\,(\mathrm{A}-e)<h\,(\mathrm{B}+e).$$

Suppose now that between the limits a and b, are interposed a series of ascending values of x, $a_1, a_2, \ldots\ldots a_n$ so near to one another that the above inequality may be always satisfied when we take one for x and the following one for $x+h$, then

$$f(a_1)-f(a)>(a_1-a)(\mathrm{A}-e)<(a_1-a)(\mathrm{B}+e)$$
$$f(a_2)-f(a_1)>(a_2-a_1)(\mathrm{A}-e)<(a_2-a_1)(\mathrm{B}+e)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f(b)-f(a_n)>(b-a_n)(\mathrm{A}-e)<(b-a_n)(\mathrm{B}+e);$$

therefore, adding,

$$f(b)-f(a)>(b-a)(\mathrm{A}-e)<(b-a)(\mathrm{B}+e);$$

or, since this is true however small e is,

$$f(b)-f(a)>(b-a)\mathrm{A}<(b-a)\,\mathrm{B}.$$

But as x changes by insensible degrees from a to b, $f'(x)$ will change by insensible degrees, and will assume all values between A and B, these being the least and greatest values

which it can have in that interval. Therefore every quantity between A and B will be a value of $f'(x)$ corresponding to some value of x between a and b. Suppose therefore $\frac{f(b)-f(a)}{b-a}$, which we have shewn to lie between A and B, to be equal to the value assumed by $f'(x)$ when $x = \lambda$, then

$$f(b) = f(a) + (b-a)f'(\lambda),$$

where λ is some quantity lying between a and b.

Newton's method of Approximation.

121. When we know an approximate value of a root, we may easily obtain other values of it, more and more exact, by a method invented by *Newton*, which rapidly attains its object. We shall give this method first in the form in which it was proposed by its author, and afterwards with the conditions which *Fourier* has shewn to be necessary for its complete success.

Let $f(x) = 0$ be an equation having a root c between a and b, the difference of these limits, $b-a$, being a small fraction whose square may be neglected in the process of approximation.

Let c_1, a quantity between a and b, be assumed as the first approximation to c, then $c = c_1 + h$, where h is very small;

$$\therefore f(c_1 + h) = 0,$$

$$\text{or } f(c_1) + f'(c_1)\,h + f''(c_1)\frac{h^2}{2} + \ldots + h^n = 0.$$

Now since h is very small, h^2, h^3, ... are very small compared with h; also none of the quantities $f''(c_1)$, $f'''(c_1)$, ... can become very great, since they result from substituting a finite value in integral functions of x; therefore, provided $f'(c_1)$ be not very small (that is, provided $f'(x) = 0$ have no root nearly equal to c_1 or to c, and consequently $f(x) = 0$ no

other root nearly equal to c besides the one we are approximating to) all the terms in the series after the two first may be neglected in comparison with them, and we have, to determine h_1 the resulting approximate value of h, the equation

$$f(c_1) + h_1 f'(c_1) = 0,$$

$$\therefore h_1 = -\frac{f(c_1)}{f'(c_1)} = -\left\{\frac{f(x)}{f'(x)}\right\}_{x=c_1},$$

and the second approximation is

$$c_2 = c_1 + h_1 = c_1 - \left\{\frac{f(x)}{f'(x)}\right\}_{x=c_1}.$$

Similarly, starting from c_2 instead of c_1, the third approximate value will be

$$c_3 = c_2 - \left\{\frac{f(x)}{f'(x)}\right\}_{x=c_2},$$

and so on; and if we can be certain that each new value is nearer to the truth than the preceding, there is no limit to the accuracy which may be obtained.

Ex. $x^3 - 2x - 5 = 0.$

Here one root lies between 2 and 3, and the equation can have only one positive root; also, upon narrowing the limits, we find that $x = 2$ gives a negative, and $x = 2{\cdot}2$ a positive result, therefore $2{\cdot}1$ differs from the root by a quantity less than $0{\cdot}1$, and we may assume $c_1 = 2{\cdot}1$. Hence

$$c_2 = 2{\cdot}1 - \left(\frac{x^3 - 2x - 5}{3x^2 - 2}\right)_{x=2{\cdot}1} = 2{\cdot}1 - \frac{0{\cdot}061}{11{\cdot}23},$$

$$\text{or } c_2 = 2{\cdot}1 - 0{\cdot}0054 = 2{\cdot}0946.$$

Similarly,

$$c_3 = 2{\cdot}09455149.$$

123. Newton's method of approximation under the following limitations is sure to succeed.

(1). The limits between which the required root is known to lie must be so close, that no other root of $f(x) = 0$, and no root of $f'(x) = 0$, or $f''(x) = 0$, lies between them.

(2). The approximation must be begun and continued from that limit which makes $f(x)$ and $f''(x)$ have the same sign.

To guard against over-correction, that is, against applying such a correction to an approximate value as shall make the new value differ more from the root by excess than the original approximate value did by defect, or *vice versâ*, we must be certain that each new value is nearer to the truth than the preceding. Now if c be a root of $f(x) = 0$ which lies between a and b, c_1 the first approximate value, and h the whole correction,

$$\text{so that } c = c_1 + h, \text{ we have } f(c_1 + h) = 0;$$

$$\therefore f(c_1) + hf'(\lambda) = 0,$$

λ being some quantity between c_1 and c, (Art. 120).

Therefore, supposing $\lambda = c_1$, which amounts to neglecting all powers of h above the first, and requires that $f(x) = 0$ have no root besides c in that interval, and calling the resulting approximate value of h, h_1, we have

$$f(c_1) + h_1 f'(c_1) = 0.$$

Now the true value is $c = c_1 + h$,

the 1st approximate value is c_1 with error h,

the 2nd . . . . $c_2 = c_1 + h_1$ with error $h - h_1$,

which (neglecting signs) must be less than h,

i. e. $h^2 - (h - h_1)^2$ must be positive, or $2hh_1 - h_1^2 = +$

$$\text{or } \frac{h}{h_1} - \tfrac{1}{2} = + \quad \text{or } \frac{f'(c_1)}{f'(\lambda)} - \tfrac{1}{2} = +,$$

which condition (since λ is an indeterminate quantity between

c_1 and c or between a and b) cannot in all cases be secured unless $f'(x)$ be incapable of changing its sign between a and b.

Moreover we must have $\dfrac{f'(c_1)}{f'(\lambda)} > \frac{1}{2}$ or > 1; that is, (since $f'(x)$ either increases or diminishes continually from a to b as $f''(x)$ preserves an invariable sign in that interval,) c_1 must be taken equal to that limit which gives $f'(x)$ its greatest numerical value without regard to sign. Hence (Art. 120) if $f'(x)$ and $f''(x)$ have the same sign from a to b, we must have $c_1 = b$, if they have contrary signs we must have $c_1 = a$. But b, which always makes $f(x)$ and $f'(x)$ have the same sign (for $f(b)$ has the same sign as $f(c + h)$ or $hf'(c)$ or $hf'(b)$), on the first supposition makes $f(x)$ and $f''(x)$ have the same sign; and a, which always makes $f(x)$ and $f'(x)$ have contrary signs, on the second supposition makes $f(x)$ and $f''(x)$ have the same sign; therefore in both cases we must have c_1 equal that limit which makes $f(x)$ and $f''(x)$ have the same sign.

These conditions being fulfilled, we have

$$\frac{f'(c_1)}{f'(\lambda)} - 1 = + \quad \text{or} \quad \frac{h - h_1}{h_1} = +,$$

$$\text{or} \quad \frac{c - c_2}{c_2 - c_1} = +;$$

therefore c_2 lies between c and c_1; hence the new limit c_2 fulfils the requisite conditions, and we may with certainty from it continue the approximation.

124. To estimate the rapidity of the approximation, we have

error in 1st approximate value c_1, $= h$

error in 2nd . . . . c_2, $= h - h_1$.

But
$$f(c_1) + hf'(c_1) + \tfrac{1}{2}h^2 f''(\mu) = 0$$
$$f(c_1) + h_1 f'(c_1) = 0;$$
$$\therefore (h - h_1) f'(c_1) + \tfrac{1}{2}h^2 f''(\mu) = 0$$
$$\text{or } h - h_1 = -\tfrac{1}{2}h^2 \cdot \frac{f''(\mu)}{f'(c_1)}.$$

Let the greatest value which $f''(x)$ can assume between a and b (which will be either $f''(a)$ or $f''(b)$, if $f'''(x) = 0$ have no root in the interval) be divided by the least value of $2f'(x)$ in that interval which will be either $2f'(a)$ or $2f'(b)$, and let the quotient be denoted by C, then, neglecting signs,

$$h - h_1 < h^2 C;$$

hence if the first error h in c_1 be a small decimal, the error $h - h_1$ with which c_2 is affected (since C will not, except in particular cases, be very large) will be very small compared with h; and if the quantity C be less than unity, the number of exact decimals in the result will be doubled by each successive operation. The quantity C, when thus computed for a given interval, preserves the same value throughout the operations which it may be necessary to make in order to approximate to the value of the root lying in that interval; and as we thus know a limit to the difference between the approximate value already found and the true value, we may always avoid calculating decimals which are inexact, and only obtain those which are necessarily correct.

Ex. $$6x^3 - 141x + 263 = 0.$$

This equation has two positive roots, one between 2·7 and 2·8, and the other between 2·8 and 2·9. Now $f'(x) = 18x^2 - 141 = 0$, has a root $= \sqrt{\frac{47}{6}} = 2{\cdot}798$ between 2·7 and 2·8, therefore these limits are not sufficiently close; but this root is greater than 2·79; also 2·7 and 2·79 substituted in $f(x)$ gives results with different signs; and 2·7 substituted in $f(x)$ and $f''(x)$ gives results with the same sign; therefore $c_1 = 2{\cdot}7$.

With regard to the other interval 2·8, 2·9, $f'(x) = 0$ $f''(x) = 0$ have no roots between these limits, and 2·9 makes $f(x)$ and $f''(x)$ have the same sign; therefore $c_1 = 2{\cdot}9$; and starting from these values we are certain in each case to get a value nearer to the truth.

Again, the greatest value which $\frac{f''(x)}{f'(2{\cdot}7)}$ can assume in the interval 2·7, 2·79 is nearly equal to 10; hence if h_1, h_2, be consecutive errors, we have $h_2 < \frac{1}{2}(h_1)^2.10$.

The same formula will be found to be true for consecutive errors in the interval 2·8, 2·9.

125. The nature of *Newton's* method of approximation, and the necessity of *Fourier's* limitations, are well illustrated by the following geometrical considerations.

Let $y = f(x)$ be the equation to a parabolic curve, then the portion of it between $x = a$ and $x = b$, (supposing these limits to satisfy all the prescribed conditions,) must have the shape PCQ, (Fig. 1,) O being the origin, OC the axis of x, PN, QM the extreme ordinates, having different signs, and there being no point of inflexion and no tangent parallel to the axis in the interval between $x = a$ and $x = b$, since neither $f''(x) = 0$, nor $f'(x) = 0$ has a root between a and b. Now if QT be a tangent at Q, it is manifest that OT will be intermediate to OC and OM, whatever be the magnitude of CM; but $\text{MT} = \frac{f(b)}{f'(b)}$ is the correction furnished by *Newton's* method; hence if we start with that end of the arc which is convex towards the axis of x, and therefore from that limit $\text{OM} = b$ which makes $f(x)$ and $f''(x)$ have the same sign, we shall get a new limit $\text{OT} = b' = b - \frac{f(b)}{f'(b)}$, which is certainly closer than the former and on the same side of the root; and if we repeat the process with b', the next value of the root will be OT′ which is still nearer to the truth. But if we commence with that end of the arc which is concave to the axis of x, and therefore from that limit $\text{ON} = a$ which makes $f(x)$ and $f''(x)$ have contrary signs, the correction will be $\text{NU} = -\frac{f(a)}{f'(a)}$, and the new value OU will

exceed OC, and may exceed OC by more than ON falls short of OC; so that we cannot be certain that the new limit is closer than the former; and if we again correct OU, the result may be still more erroneous.

We may however obtain a new inferior limit by drawing PS parallel to QT, then OS will always lie between ON and OC, and we have NS $= -\dfrac{f(a)}{f'(b)}$ and OS $= a - \dfrac{f(a)}{f'(b)}$. Thus we have two new limits, and as many figures as their values have in common, so many are exact in the approximation.

If the primitive interval were not sufficiently small to exclude all roots of $f'(x) = 0$ and $f''(x) = 0$, then it might happen that the limit b might correspond to a point B situated beyond a point of inflexion R, and the tangent at B might meet the axis at a point remote from C; and if B were situated at the extremity of a maximum ordinate, the result would be still more erroneous.

Lagrange's Method of Approximation by continued Fractions.

126. Before proceeding to the main object of finding the roots of equations under the forms of continued fractions, it will be necessary to investigate several general properties of that sort of expressions.

Every expression having the form

$$a + \cfrac{\beta}{b + \cfrac{\gamma}{c + \cfrac{\delta}{d + \dots}}}$$

is called a continued fraction. We shall at present consider only the case where the numerators $\beta, \gamma, \delta, \dots$ are equal

to unity, and the quantities $a, b, c, \ldots\ldots$ are positive integers, so that the continued fraction will be of the form

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \ldots}}} \qquad \text{or } a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+\ldots}$$

as it may be conveniently written.

127. Expressions of this sort present themselves, whenever we attempt to express numerically the values of fractional or irrational quantities. For suppose we were required to estimate the value of a quantity x, not expressible by an integer; if we first seek the whole number a which is next less than x, the difference $x - a$ is a fraction less than unity, which we may represent by $\frac{1}{y}$, y being a quantity greater than unity; similarly, if b be the whole number next less than y, the difference $y - b$ may be represented by $\frac{1}{z}$, z being a quantity greater than unity. Proceeding in this manner, we have

$$x = a + \frac{1}{y},\ y = b + \frac{1}{z},\ z = c + \frac{1}{u},\ u = d + \frac{1}{v}, \ldots\ldots$$

$$\therefore x = a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+\ldots}.$$

If among the quantities $x, y, z, \ldots$ there occurs one which is exactly expressible by an integer, the continued fraction terminates; in the contrary case, it may be prolonged indefinitely. The former, as we proceed to show, will happen whenever the quantity proposed to be transformed is a commensurable fraction; and the latter, when it is irrational or otherwise incommensurable; and the corresponding limited and unlimited continued fractions are called rational and irrational respectively.

128. To convert any proposed fraction $\frac{m}{n}$ into a continued fraction.

The integer next less than $\frac{m}{n}$ is the quotient of the division of m by n, let a be the quotient and p the remainder, then

$$\frac{m}{n} = a + \frac{p}{n}.$$

Similarly, let b be the quotient of the division of n by p, and q the remainder, then

$$\frac{n}{p} = b + \frac{q}{p}.$$

Again,

$$\frac{p}{q} = c + \frac{r}{q}, \ldots .$$

$$\therefore \frac{m}{n} = a + \frac{1}{b+} \frac{1}{c + \ldots .}$$

Hence we see that, to reduce a vulgar fraction to a continued fraction, we must proceed exactly in the same manner as to find the greatest common measure of its numerator and denominator; taking care, however, first to divide the numerator by the denominator, so that when the numerator is less than the denominator, the first quotient, a, will be zero. And as the process of finding the greatest common measure of two numbers always leads to a remainder zero, and a quotient expressed exactly by an integer, we see that every commensurable quantity can be expressed by a continued fraction which terminates; and conversely, every terminating continued fraction is the expression of a commensurable quantity, for by performing the calculations indicated, it can be reduced to an ordinary fraction.

Ex. By performing the process of finding the greatest common measure of 743 and 611, we find the quotients 1, 4, 1, 1, 1, 2, 3, 1, 3, and a remainder zero;

$$\therefore \frac{743}{611} = 1 + \frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{3}.$$

Hence also it results that an incommensurable quantity can be converted only into a continued fraction which does not terminate.

129. To convert $\sqrt{N}$ (N not being a complete square) into a continued fraction.

Let a^2 be the greatest square in N, so that $N = a^2 + b$; then a is the greatest integer in $\sqrt{N}$, let $\frac{1}{x'}$ be the remainder;

$$\therefore x = \sqrt{N} = a + \frac{1}{x'},$$

$$x' = \frac{1}{\sqrt{N} - a} = \frac{\sqrt{N} + a}{b} = a + \frac{1}{x''},$$

a being the greatest integer in x' and $\frac{1}{x''}$ the remainder. Suppose that in continuing this process we arrive at

$x^{(r)} = \dfrac{\sqrt{N} + m}{n} = \mu + \dfrac{1}{y}$; μ being the greatest integer in $x^{(r)}$,

$$\therefore y = \frac{n}{\sqrt{N} - (n\mu - m)} = \frac{n(\sqrt{N} + n\mu - m)}{N - (n\mu - m)^2}$$

$$= \frac{\sqrt{N} + m'}{n'}$$

if $m' = n\mu - m$, $n' = \dfrac{N - (n\mu - m)^2}{n} = \dfrac{N - m'^2}{n'}$;

$\therefore y = \dfrac{\sqrt{N} + m'}{n'} = \mu' + \dfrac{1}{z}$, μ' being the greatest integer in y.

Similarly, $z = \dfrac{\sqrt{N} + m''}{n''} = \mu'' + \dfrac{1}{u}$, ...

m'', n'' being formed from m', n', exactly in the same manner as m', n' were from m and n, and μ'' being the nearest integer to z, and so on for the rest. Hence y, z, ... and the quotients μ', μ'', ... will be found by an easy and uniform process, which must be continued till we arrive at a quotient $= 2a$; after which, the quotients will recur (as will be hereafter shewn) in the same order, beginning with a.

Since $m' = n\mu - m$, and $nn' = N - (n\mu - m)^2$

or $n' = n^\circ + 2\mu m - n\mu^2$, since $nn^\circ = N - m^2$

$\left(\frac{\sqrt{N} + m^\circ}{n^\circ}\right.$ being the quantity which precedes $\left.\frac{\sqrt{N} + m}{n}\right)$, we see that m and n will always be integers, since they are so in the two first cases $\frac{\sqrt{N} + 0}{1}$ and $\frac{\sqrt{N} + a}{b}$, and it will appear that they are always positive.

Ex. To express $\sqrt{23}$ by a continued fraction.

$\frac{\sqrt{23}+0}{1} = 4$, writing down only the integral part;

also $m = 0$, $n = 1$, $\therefore m' = 4.1 - 0 = 4$, $n' = \frac{23-16}{1} = 7$,

$$\frac{\sqrt{23}+4}{7} = 1 \quad m'' = 7.1 - 4 = 3, n'' = \frac{23-9}{7} = 2,$$

$$\frac{\sqrt{23}+3}{2} = 3 \quad m''' = 2.3 - 3 = 3, n''' = \frac{23-9}{2} = 7,$$

$$\frac{\sqrt{23}+3}{7} = 1 \quad m^{iv} = 7.1 - 3 = 4, n^{iv} = \frac{23-16}{7} = 1,$$

$$\frac{\sqrt{23}+4}{1} = 8 \quad m^{v} = 1.8 - 4 = 4, n^{v} = \frac{23-16}{1} = 7,$$

$$\frac{\sqrt{23}+4}{7} = 1.$$

Hence the quotients 1, 3, 1, 8 will recur, and

$$\sqrt{23} = 4 + \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{1+}\,\frac{1}{3+\ldots}.$$

130. Returning to the consideration of the expression

$$x = a + \frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{d+\ldots},$$

the fractions formed by taking 1, 2, 3, ... of the quantities $a, b, c, \ldots.$, are called converging fractions; thus

$$\frac{a}{1},\ a + \frac{1}{b} = \frac{ab+1}{b},\ a + \frac{1}{b + \frac{1}{c}} = \frac{abc + a + c}{bc + 1}, \ldots.$$

are converging fractions.

The converging fractions taken in order are alternately less and greater than the true value of x; thus $\frac{a}{1}$ is too small; $a + \frac{1}{b}$ is too large, because a part of the denominator is omitted; again $b + \frac{1}{c}$ is too large, and therefore $a + \frac{1}{b + \frac{1}{c}}$ is too small, and so on.

The quantities $a, b, c, \ldots.$ are called quotients, and any one with the quantity which must be added to it (supposing it were the last) to make the value of x exact, is called a complete quotient. Thus, using the notation of Art. 127,

$$a + \frac{1}{y},\ b + \frac{1}{z},\ c + \frac{1}{u}, \ldots.$$

are complete quotients.

131. To transform any continued fraction into a series of converging fractions.

Suppose $\frac{p}{q}, \frac{p'}{q'}, \frac{p''}{q''}$, three successive converging fractions.

Write down the quotients, and under them the converging fractions,

$$\begin{array}{cccccccc} a & b & c & d\ldots & & m & m' & m'' \\ & \frac{a}{1} & \frac{ab+1}{b} & \frac{abc+a+c}{bc+1} & \ldots & \frac{p}{q} & \frac{p'}{q'} & \frac{p''}{q''}. \end{array}$$

Now as far as we have gone we observe, that, having formed the first two converging fractions, and written them one row in advance of the quotients, the numerator of any fraction is formed by multiplying the numerator of the preceding by the quotient that stands over it, and adding the numerator of the fraction preceding that, thus

$$abc + a + c = (ab + 1)\,c + a;$$

and the denominator, in the same manner, by multiplying the denominator of the preceding by the quotient over it, and adding the denominator preceding that,

$$\text{thus } bc + 1 = bc + \text{the denominator of } \frac{a}{1}.$$

Suppose the law to hold up to the quotient m, so that $\left(\frac{p^\circ}{q^\circ}\right.$ being the fraction preceding $\left.\frac{p}{q}\right)$

$$p' = pm + p^\circ, \quad q' = qm + q^\circ,$$

$$\text{then } \frac{p'}{q'} = \frac{pm + p^\circ}{qm + q^\circ}.$$

Now $\frac{p''}{q''}$ differs from $\frac{p'}{q'}$ only in taking in another quotient, so that if $m + \frac{1}{m'}$ be written for m, we have

$$\frac{p''}{q''} = \frac{p\left(m + \frac{1}{m'}\right) + p^\circ}{q\left(m + \frac{1}{m'}\right) + q^\circ} = \frac{(pm + p^\circ)\,m' + p}{(qm + q^\circ)\,m' + q} = \frac{p'm' + p}{q'm' + q},$$

which is the same form as the preceding; if therefore the law hold for the formation of any one converging fraction, it holds for the formation of the next; but we have seen that it holds for the third, therefore the law obtains generally.

Ex. 1. To find a series of fractions converging to $\frac{743}{611}$.

Here the quotients are (Art. 128) 1, 4, 1, 1, 1, 2, 3, 1, 3, and the two first fractions are $\frac{1}{1}$, and $1+\frac{1}{4}$ or $\frac{5}{4}$.

Hence, writing down the quotients and the two first fractions in the manner directed above, and forming the rest by the rule, we have

$$\begin{array}{ccccccccc} 1 & 4 & 1 & 1 & 1 & 2 & 3 & 1 & 3 \\ \frac{1}{1} & \frac{5}{4} & \frac{6}{5} & \frac{11}{9} & \frac{17}{14} & \frac{45}{37} & \frac{152}{125} & \frac{197}{162} & \frac{743}{611}, \end{array}$$

the last being the original fraction, and the preceding alternately greater and less than the true value.

If the proposed quantity has no integral part, then the first quotient, as was before observed, will be zero, and the first converging fraction $\frac{0}{1}$.

Ex. 2. To find a series of fractions converging to $\sqrt{23}$.

The quotients are (Art. 129) 4, 1, 3, 1, 8, 1, 3, 1, 8, . . .

Hence we have

$$\begin{array}{ccccccccc} 4 & 1 & 3 & 1 & 8 & 1 & 3 & 1 & 8 \ldots\ldots \\ \frac{4}{1} & \frac{5}{1} & \frac{19}{4} & \frac{24}{5} & \frac{211}{44} & \frac{235}{49} & \frac{916}{191} & \frac{1151}{240} & \ldots\ldots \end{array}$$

132. The difference between any two consecutive converging fractions, is a fraction whose numerator is unity, and denominator the product of the denominators of the fractions.

This is immediately verified with respect to the two first converging fractions, for

$$\frac{ab+1}{b} - \frac{a}{1} = \frac{1}{b}.$$

To prove therefore that it is generally true, it will be sufficient to consider three consecutive fractions $\frac{p^\circ}{q^\circ}, \frac{p}{q}, \frac{p'}{q'}$, and to shew that if the property holds for the two $\frac{p^\circ}{q^\circ}, \frac{p}{q}$, it must hold also for $\frac{p}{q}$ and $\frac{p'}{q'}$.

$$\text{Now } \frac{p'}{q'} - \frac{p}{q} = \frac{mp+p^\circ}{mq+q^\circ} - \frac{p}{q} = \frac{qp^\circ - pq^\circ}{(mq+q^\circ)\,q},$$

and by the hypothesis

$$\frac{p}{q} - \frac{p^\circ}{q^\circ} = \pm\frac{1}{qq^\circ} \text{ or } pq^\circ - qp^\circ = \pm 1;$$

$$\therefore \frac{p'}{q'} \sim \frac{p}{q} = \frac{1}{q'q} \text{ or } p'q \sim q'p = 1.$$

133. Every converging fraction approaches nearer to the true value of x than that which precedes it; and (the integral part, or zero, being the first converging fraction) all the converging fractions of an odd order are less, and all those of an even order greater, than the true value.

$$\text{We have } \frac{p'}{q'} = \frac{mp+p^\circ}{mq+q^\circ},$$

and to deduce the value of x from that of $\frac{p'}{q'}$, it is sufficient to replace the quotient m by the complete quotient $m + \frac{1}{\mu} = y$ suppose, where y is always positive and greater than unity;

$$\therefore x = \frac{py+p^\circ}{qy+q^\circ};$$

$$\therefore x - \frac{p}{q} = \pm \frac{1}{q(qy + q^\circ)}, \quad \frac{p^\circ}{q^\circ} - x = \pm \frac{y}{q^\circ(qy + q^\circ)}.$$

Now $q^\circ < q$ and $y > 1$, therefore on both accounts the value of $x - \frac{p}{q}$ is less than the value of $\frac{p^\circ}{q^\circ} - x$ (not regarding the signs). Therefore the successive converging fractions approach nearer and nearer to x; and since $\frac{p^\circ}{q^\circ} - x$ and $x - \frac{p}{q}$ have the same sign, the successive converging fractions are alternately greater and less than the true value; but the first converging fraction $\frac{a}{1}$ is less than x; therefore all the converging fractions of an odd order are less than x, and form an increasing series; and all the converging fractions of an even order are greater than x, and form a decreasing series.

134. All converging fractions are in their lowest terms.

For if the numerator and denominator of the fraction $\frac{p}{q}$ had a common measure, then from the equation $p'q - q'p = \pm 1$, it would follow that this common factor must divide unity.

135. The error, in taking any converging fraction for the value of the continued fraction, is less than unity divided by the product of the denominators of that fraction and the following one; and greater than unity divided by the product of that denominator and the sum of that denominator and the following one.

$$\text{For since } x - \frac{p}{q} = \pm \frac{1}{q(qy + q^\circ)},$$

and y is greater than m and less than $m + 1$; therefore, leaving the sign out of consideration,

$$x - \frac{p}{q} < \frac{1}{q(qm + q^\circ)} > \frac{1}{q(qm + q + q^\circ)};$$

or, since $q' = qm + q^{\circ}$,

$$x - \frac{p}{q} < \frac{1}{qq'} > \frac{1}{q(q' + q)}.$$

136. We can also obtain a superior limit of the error depending only on the denominator of that converging fraction which we take for the approximate value; and an inferior limit depending only on the denominator of the following one. For since q' is always greater than q, $\frac{1}{qq'}$ is less than $\frac{1}{q^2}$, and $\frac{1}{q(q'+q)}$ greater than $\frac{1}{2q'^2}$; therefore, *à fortiori*,

$$x - \frac{p}{q} < \frac{1}{q^2} > \frac{1}{2q'^2}.$$

These limits are to be preferred on account of their simplicity, and in most cases are sufficiently exact.

Hence we may at any step measure the accuracy of our approximation. Thus, in the examples of Art. 131, the fraction $\frac{152}{125}$, which converges towards $\frac{743}{611}$, differs from it by a quantity less than $\frac{1}{(125)^2}$ and greater than $\frac{1}{2(162)^2}$; and the fraction $\frac{916}{191}$, which converges towards $\sqrt{23}$, differs from it by a quantity less than $\frac{1}{(191)^2}$, and greater than $\frac{1}{2(240)^2}$.

Ex. To find a series of fractions converging to the value of the ratio of the circumference of a circle to its diameter; and to estimate the error with which each is affected.

The value of this ratio, exact to ten places of decimals, is 3·1415926535; therefore, adding unity to the last decimal, the value of π will be comprised between the fractions

$$\frac{31\,415\,926\,535}{10\,000\,000\,000} \text{ and } \frac{31\,415\,926\,536}{10\,000\,000\,000}.$$

If now we perform the successive divisions for each fraction, we find the two series of quotients

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad 1, \quad 6$$

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad 1, \quad 1;$$

therefore reserving only the quotients which are common to both, and which must belong to the continued fraction which expresses the value of π, and forming the converging fractions, we have

$$\begin{array}{ccccccc} 3, & 7, & 15, & 1, & 292, & 1, & 1, \\ & \frac{3}{1}, & \frac{22}{7}, & \frac{333}{106}, & \frac{355}{113}, & \frac{103993}{33102}, & \frac{104348}{33215}. \end{array}$$

These fractions are alternately greater and less than the true value of π; thus $\frac{22}{7}$ is too great; it is the ratio discovered by *Archimedes*, and differs from the true value by a quantity lying between $\frac{1}{7 \times 106}$ and $\frac{1}{7(7 + 106)}$ or $\frac{1}{742}$ and $\frac{1}{791}$. The fraction $\frac{355}{113}$ is the value discovered by *Adrien Métius*; it is also too great, but far nearer than that of *Archimedes*, since it only leaves an error comprised between $\frac{1}{3740526}$ and $\frac{1}{3753295}$.

137. In order that the fraction $\frac{p}{q}$ may differ from the exact value of x by a quantity less than a given quantity $\frac{1}{a}$, it is sufficient that we have $\frac{1}{q^2} < \frac{1}{a}$, or $q =$ or $> \sqrt{a}$.

Hence we can always obtain, either exactly, or within any degree of approximation, the value of a quantity expressed by a continued fraction; for if the continued fraction terminates, we then obtain its value exactly; and if it does not terminate, we can obtain a converging fraction whose denominator

satisfies the condition $q =$ or $> \sqrt{a}$, because the denominators of the converging fractions are integers, and go on increasing.

138. Any converging fraction, $\frac{p}{q}$, expresses the value of the continued fraction, more exactly than any fraction $\frac{r}{s}$ whose denominator is less than q.

If $\frac{r}{s}$ be one of the converging fractions, this is manifest from what has been proved. But if $\frac{r}{s}$ be not one of the converging fractions, then it cannot lie between $\frac{p}{q}$ and the preceding $\frac{p^{\circ}}{q^{\circ}}$; for if it could, then $\pm\left(\frac{p^{\circ}}{q^{\circ}} \quad \frac{r}{s}\right)$ would be less than $\frac{1}{q^{\circ}q}$ or $\pm\left(p^{\circ}s - q^{\circ}r\right) < \frac{s}{q}$, which is impossible, because the first member of this inequality is an integer different from zero, and the second a proper fraction, since $s < q$.

Since then $\frac{r}{s}$ cannot lie between $\frac{p^{\circ}}{q^{\circ}}$ and $\frac{p}{q}$, if it lie to the right of $\frac{p}{q}$ (supposing the three arranged in order of magnitude) it differs from x more than $\frac{p}{q}$ does; and if it lie to the left of $\frac{p^{\circ}}{q^{\circ}}$, it differs from x more than $\frac{p^{\circ}}{q^{\circ}}$ does, and therefore *à fortiori* more than $\frac{p}{q}$ does.

139. Every periodic continued fraction expresses one of the roots of a quadratic equation whose coefficients are commensurable.

Let the continued fraction be

$$x = a + \frac{1}{b+} \ldots \frac{1}{k+} \frac{1}{l+} \frac{1}{y},$$

$$\text{where } y = r + \frac{1}{s+} \ldots \frac{1}{u+} \frac{1}{v+} \frac{1}{y};$$

so that $a, b, c, \ldots . l$ are quotients which do not recur, and $r, s, \ldots . v$ are those which recur indefinitely.

Let $\frac{p}{q}, \frac{p'}{q'}$, be converging fractions in the value of x, the last quotients respectively comprised in them being k and l, so that l and r are the quotients which stand over them, when formed according to the method of Art. 131; and $\frac{\text{P}}{\text{Q}}, \frac{\text{P}'}{\text{Q}'}$, those in the value of y, the last quotients respectively comprised in them being u and v; then, as in Art. 133,

$$x = \frac{p'y + p}{q'y + q}, \; y = \frac{\text{P}'y + \text{P}}{\text{Q}'y + \text{Q}},$$

between which equations, eliminating y, we obtain an equation of the second degree in x, which demonstrates the property announced. When we wish to find x under an irrational form, we must take the positive value of y in the equation

$$\text{Q}'y^2 + (\text{Q} - \text{P}')\, y - \text{P} = 0,$$

and substitute it in the preceding value of x.

140. To approximate to the roots of an equation by the method of continued fractions.

Let the equation $f(x) = 0$ have only one root between the integers a and $a + 1$, then writing $a + \frac{1}{y}$ for x, the first transformed equation will be

$$f(a) + f'(a)\frac{1}{y} + f''(a)\frac{1}{1.2y^2} + \ldots . + \frac{1}{y^n} = 0 \;(1),$$

and since $\frac{1}{y}$ lies between 0 and 1, y has only one value greater

than 1; if therefore we substitute successively 2, 3, 4, ... for y, stopping at the first which gives a positive result, the integer preceding that, is the integral part of the value of y. Let this be b, and in (1) write $b + \frac{1}{z}$ for y; then the second transformed equation will have only one root greater than unity, the integral part of which, as before, will be the whole number next less than the one in the series 2, 3, 4, ... which first gives a positive result when written for z; let this be c, and in the second transformed equation write $c + \frac{1}{u}$ for z, then the third transformed equation will have only one root greater than unity, the integral part of which may be found as before, and so on. We thus obtain successively the terms of a continued fraction

$$a + \frac{1}{b +} \frac{1}{c +} \frac{1}{d + \ldots},$$

which expresses the required value of x; consequently we are able (Art. 137) to find this value to any required degree of exactness. If any of the numbers $b, c, d, \ldots$ is an exact root of the corresponding transformed equation, the process terminates, and we find the exact value of x. Also, if one of the transformed equations be identical with a preceding one, the continued fraction expressing the root is periodical; for, after that, the same quotients will recur in the same order; in this case a finite value, in the form of a surd, may be obtained for the root (Art. 139) by solving a quadratic whose coefficients are rational, both of whose roots will be roots of the proposed, (Art. 15) since the coefficients of the latter are supposed rational; consequently the first member of this quadratic will be a factor of the first member of the proposed equation, which may therefore be depressed two dimensions.

Ex. To find the positive root of $x^3 - 2x - 5 = 0$ under the form of a continued fraction.

Comparing this with $x^3 - qx + r = 0$, we find that

$$\frac{r^2}{4} - \frac{q^3}{27} = \frac{25}{4} - \frac{8}{27} \text{ is a positive quantity,}$$

therefore the equation has two impossible roots; and since its last term is negative, its third root is positive. Substituting 2 and 3, the results are -1 and $+16$, therefore the root lies between 2 and 3. Assume $x = 2 + \frac{1}{y}$, and the transformed equation is

$$y^3 - 10y^2 - 6y - 1 = 0,$$

in which 10 and 11 being substituted give -61, $+54$. Assume $y = 10 + \frac{1}{z}$, and we obtain

$$61z^3 - 94z^2 - 20z - 1 = 0$$

whose root lies between 1 and 2. Proceeding in this manner we find

$$x = 2 + \frac{1}{10+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+\ldots}$$

the value of the root, in a continued fraction, which may be converted into a series of converging fractions.

141. When an equation has several roots between two consecutive integers, this method of approximating to them may be rendered easier by combining it with *Sturm's* theorem.

Substituting 0, 1, 2, 3, ... successively for x in the series of quantities (Art. 101)

$$f(x),\ f_1(x),\ f_2(x), \ldots . f_n(x) \ldots (1)$$

and noting between what substitutions, changes of sign are lost, and how many, we shall perceive between what integers the

roots lie, and how many in each interval. For those roots which are situated singly between consecutive integers, the process will be that described above; but for those which lie in groups between consecutive integers, we must proceed as follows. Suppose several roots to lie between a and $a + 1$; substitute $a + \frac{1}{y}$ for x in series (1), and let the result be

$$\phi(y), \quad \phi_1(y), \quad \phi_2(y), \ldots \phi_n(y) \ldots (2),$$

then as many roots as $f(x) = 0$ has between a and $a + 1$, so many positive roots greater than unity will $\phi(y) = 0$ have; and if we write $1, 2, 3, \ldots$ for y in series (2), and observe between what substitutions changes are introduced, the consecutive integers between which those values of y, either singly or in groups, are situated, will be determined. If there still be a group of values of y between consecutive integers b and $b + 1$, put $y = b + \frac{1}{z}$ in series (2), and let the result be

$$\psi(z), \quad \psi_1(z), \quad \psi_2(z), \ldots \psi_n(z) \ldots (3),$$

then as many roots as $\phi(y) = 0$ has between b and $b+1$, so many positive roots greater than unity will $\psi(z) = 0$ have; and, as before, substituting $1, 2, 3, \ldots$ for z in series (3), and observing where the changes are introduced, we may determine the situation of those values of z; and the process must be continued till we arrive at a transformed equation whose positive roots are situated singly between consecutive integers; the approximation to each of these roots, as well as to all those already partially found, may then be continued, as in Art. 140, to any degree of accuracy. Thus all the values of x between a and $a + 1$ will be determined; and the other groups of values of x, if there be any, must be treated in the same manner.

142. It is manifest that *Fourier's* method of separating the

roots might be employed with similar advantages. For being applied to the proposed equation $f(x) = 0$, it would enable us to ascertain between what integers the roots lie, and how many in each interval. Next being applied to the transformed equation $f\left(a + \frac{1}{y}\right) = 0$, or $\phi(y) = 0$, a and $a + 1$ being one of those intervals, it would point out between what integers the values of y lie, and how many in each interval. Similarly in the next transformed equation $\phi\left(b + \frac{1}{z}\right) = 0$ or $\psi(z) = 0$, b and $b + 1$ being an interval containing more values of y than one, it would shew the situation of the values of z; and so on for all the transformed equations which it might be necessary to obtain, to completely separate the process for approximating to each root of $f(x) = 0$.

The following is an instance of the employment of *Sturm's* theorem.

Ex.
$$f(x) = 6x^3 - 141x + 263$$
$$f_1(x) = 6x^2 - 47$$
$$f_2(x) = 94x - 263$$
$$f_3(x) = +$$

	f	f_1	f_2	f_3
(2)	+	−	+	+
(3)	+	+	+	+ ;

hence two values of x lie between 2 and 3;

$$\therefore\ x = 2 + \frac{1}{y}$$

$$\phi(y) = 29y^3 - 69y^2 + 36y + 6$$
$$\phi_1(y) = -23y^2 + 24y + 6$$
$$\phi_2(y) = -75y + 94$$
$$\phi_3(y) = +$$

	ϕ	ϕ_1	ϕ_2	ϕ_3
(1)	+	+	+	+
(2)	−	−	−	+
(3)	+	−	−	+.

Hence one value of y lies between 1 and 2, and the other between 2 and 3, and the roots must now be approximated to by separate processes.

143. Another useful application of continued fractions is to find the integral values of x and y, which satisfy the indeterminate equation of the first order $ax + by = c$.

We suppose that a, b, c are integers positive or negative, and the two former prime to one another, for if they are not, c must necessarily have the same divisor, since x and y represent integers. Let $x = \alpha$, $y = \beta$ be a solution, then

$$a\alpha + b\beta = c,$$

and therefore by subtraction, $a(x - \alpha) = - b(y - \beta)$;

but since a and b are prime to one another, $y - \beta$ must be a multiple of $a = at$ suppose; therefore $x - \alpha = - bt$,

$$\text{that is, } x = \alpha - bt,\ y = \beta + at,$$

where t is any integer positive or negative. Now to find α and β, resolve $\frac{a}{b}$ into a continued fraction; and in the series of converging fractions, let $\frac{p}{q}$ be that which immediately precedes $\frac{a}{b}$, then $pb - qa = \pm 1$, according as $\frac{p}{q} >$ or $< \frac{a}{b}$;

$$\therefore cp \cdot b - cq \cdot a = \pm c;$$

hence, comparing this with the proposed equation, if the second members have the same sign, $\beta = cp$, $\alpha = - cq$; if different signs, $\beta = - cp$, $\alpha = cq$.

Ex. $$5x + 7y = 29.$$

$\frac{7}{5} = 1 + \frac{1}{2+}\frac{1}{2}$, and the converging fractions are $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}$;

$$\therefore\ 3.5 - 7.2 = 1, \text{ and } 5.87 - 7.58 = 29;$$

$$\therefore\ x = 87 - 7t, \quad y = -58 + 5t.$$

144. When we wish to solve $ax + by = c$ in positive integers, t must be restricted in the general values $x = \alpha - bt$, $y = \beta + at$.

First, suppose a and b to be positive, and therefore c positive since x and y are to be positive; then we must have $\alpha - bt > 0$, $\beta + at > 0$, or $t < \frac{\alpha}{b}$ and $> -\frac{\beta}{a}$; therefore only those integral values of t which are comprised between the limits $-\frac{\beta}{a}, \frac{\alpha}{b}$ are admissible. These limits are never contradictory; for since α and β are positive or negative integers which satisfy the relation $a\alpha + b\beta = c$, we have $a\alpha + b\beta > 0$, and $\therefore \frac{\alpha}{b} > -\frac{\beta}{a}$; but they may not include any integer, in which case the proposed equation has no solution in integers; and in no case has it more than a certain number of such solutions.

Secondly, let the equation be $ax - by = c$, a and b being positive; then $x = \alpha + bt$, $y = \beta + at$; and in order that these values may be positive, we must have $t > -\frac{\alpha}{b}$ and $t > -\frac{\beta}{a}$; hence we may give t any value above the greatest of these limits, so that the proposed equation will admit of an infinite number of solutions in positive integers.

In the Ex. (Art. 143) $t < 12\frac{3}{7} > 11\frac{3}{5}$; $\therefore$ t has only one value 12; and $x = 3$, $y = 2$ are the only positive integral values.

145. The last application we shall make of continued fractions shall be to determine the nature of the development of the square root of a number not a complete square, in that form; preparatory to which the following property must be demonstrated.

Let $\frac{1}{a+}\ \frac{1}{b+}\ \frac{1}{c+\ ..}\ \frac{1}{m+}\ \frac{1}{m'+}\ \frac{1}{m''}$ be the development of a proper fraction $\frac{\text{P}}{\text{Q}}$; then writing down the quotients and corresponding converging fractions, we have

$$a,\quad b,\quad c,\ \ldots\quad m,\ m',\ m''.$$
$$\frac{1}{a}\ \frac{b}{ab+1}\ \cdots\ \frac{p}{q}\ \frac{p'}{q'}\ \frac{p''}{q''}\ \frac{\text{P}}{\text{Q}},$$

whence we obtain the following equations:

$$\text{Q} = m''q'' + q', \quad \therefore \frac{q''}{\text{Q}} = \cfrac{1}{m'' + \cfrac{q'}{q''}}$$

$$q'' = m'q' + q, \quad \therefore \frac{q'}{q''} = \cfrac{1}{m' + \cfrac{q}{q'}}$$

$$q' = mq + q^{\circ}, \quad \therefore \frac{q}{q'} = \cfrac{1}{m + \cfrac{q^{\circ}}{q}}, \ \ldots$$

$$\therefore \frac{q''}{\text{Q}} = \frac{1}{m''+}\ \frac{1}{m'+}\ \frac{1}{m+\ ..}\ \frac{1}{b+}\ \frac{1}{a};$$

that is, the development of $\frac{q''}{\text{Q}}$ in a continued fraction $\left(\frac{p''}{q''}\right.$ being the last of the series of fractions which converge to $\left.\frac{\text{P}}{\text{Q}}\right)$ gives the same quotients as the development of $\frac{\text{P}}{\text{Q}}$, but in an inverted order; if therefore in any case $q'' = \text{P}$, the series of quotients will be symmetrical, *i.e.* the same taken from the beginning and end, or of the form $a, b, c\ \ldots\ c, b, a$.

146. If N be a whole number (not a complete square), then $\sqrt{N}$ may be developed in an indefinite continued fraction whose quotients recur in periods, the last quotient in each period being double of the greatest number whose square is less than N, and the period, as to the other quotients, being the same taken from the beginning and end.

In the continued fraction which expresses $\sqrt{N}$ (formed as explained in Art. 129), let the series of complete quotients, partial quotients, and converging fractions, be

$$
\begin{array}{ccccccc}
\sqrt{N}, & \frac{\sqrt{N}+a}{b}, \ldots & \frac{\sqrt{N}+m^\circ}{n^\circ}; & \frac{\sqrt{N}+m}{n}, & \frac{\sqrt{N}+m'}{n'}, \ldots & \frac{\sqrt{N}+m_1{}^\circ}{n_1{}^\circ}; & \frac{\sqrt{N}+m}{n}, \ldots \\
a, & a, \ldots & \mu^\circ; & \mu, & \mu', \ldots & \omega; & \mu, \ldots \\
 & \frac{a}{1}, \ldots & \frac{p^\circ}{q^\circ}; & \frac{p}{q}, & \frac{p'}{q'}, \ldots & \frac{p_1{}^\circ}{q_1{}^\circ}; & \frac{p_1}{q_1}, \ldots
\end{array}
$$

then any complete quotient $\frac{\sqrt{N}+m}{n}$ is formed from that which precedes it by the law $m = \mu^\circ n^\circ - m^\circ$, $n = \frac{N - m^2}{n^\circ}$; and we must first shew that all the quantities $m, n, m', n', \ldots$ are positive integers. Suppose this to be the case up to m°, n°, then all the partial quotients up to μ° are positive integers, and the converging fractions up to $\frac{p}{q}$ inclusive can be formed in the usual way; and therefore, since $\frac{\sqrt{N}+m}{n}$ is the complete quotient corresponding to $\frac{p}{q}$, we have

$$\sqrt{N} = \frac{p\left(\frac{\sqrt{N}+m}{n}\right) + p^\circ}{q\left(\frac{\sqrt{N}+m}{n}\right) + q^\circ},$$

which, by equating rational and irrational parts, gives

$$\begin{aligned} pm + p^\circ n &= q\mathrm{N} \\ qm + q^\circ n &= p \end{aligned} \quad \text{or} \quad \begin{cases} (pq^\circ - qp^\circ)\, m = qq^\circ \mathrm{N} - pp^\circ \\ (pq^\circ - qp^\circ)\, n = p^2 - \mathrm{N}q^2. \end{cases}$$

But $pq^\circ - qp^\circ = +1$ or -1, according as $\frac{p}{q} >$ or $< \sqrt{\mathrm{N}}$, therefore n is a positive integer; also the equation $qm + q^\circ n = p$ gives $\frac{q^\circ}{q} = \frac{1}{n}\left(\frac{p}{q} - m\right)$; and since $q > q^\circ$, $n > \frac{p}{q} - m$, and consequently $n > \sqrt{\mathrm{N}} - m$; but $\frac{\sqrt{\mathrm{N}} + m}{n} > \mu$, $\therefore n < \sqrt{\mathrm{N}} + m$, which would be impossible if m were negative. Hence m and n will be always positive integers, since they are so in the two first cases.

We can now find the limits which m and n cannot surpass, however far the process be carried on; for the equation $\mathrm{N} - m^2 = nn^\circ$ shews that $m < \sqrt{\mathrm{N}}$, and therefore m cannot exceed a the nearest integer to $\sqrt{\mathrm{N}}$; and since $m + m^\circ = \mu^\circ n^\circ$, $2a$ is the limit both of n° and μ°. But since the continued fraction which expresses $\sqrt{\mathrm{N}}$ is unlimited, and since there can only be a certain number of values of m and n, the same value of m must occur with the same value of n an infinite number of times, that is, the same complete quotient must recur; and whenever this happens, then the succeeding quotients will be the same as those before obtained, and will recur in the same order; therefore the continued fraction which expresses $\sqrt{\mathrm{N}}$ will (at least after a certain number of terms) be composed of a constant period of quotients, and we must now determine the point at which that period begins.

Suppose the recurring period of quotients to be

$$\mu, \mu', \mu'', \ldots \omega;$$

then since $\mathrm{N}^2 - m^2 = nn^\circ$, and $\mathrm{N}^2 - m^2 = nn_1^\circ$, $\therefore n^\circ = n_1^\circ$;

also since $m = \mu^\circ n^\circ - m^\circ$, $\quad m = \omega n_1{}^\circ - m_1{}^\circ$,

$$\therefore\ m^\circ - m_1{}^\circ = n^\circ(\mu^\circ - \omega).$$

But the equation

$$qm + q^\circ n = p \text{ gives } m = \frac{p}{q} - \frac{q^\circ}{q}n = a + \frac{r}{q} - \frac{q^\circ}{q}n,$$

since $\frac{p}{q}$, being an approximate value of $\sqrt{\mathrm{N}}$, can only differ from a by a small fraction $\frac{r}{q}$;

$$\therefore\ a - m = n\frac{q^\circ}{q} - \frac{r}{q};$$

therefore, since $q^\circ < q$, $a - m < n$,

hence $a - m^\circ < n^\circ$ and $a - m_1{}^\circ < n^\circ$;

therefore $m^\circ - m_1{}^\circ < n^\circ$ or $\dfrac{m^\circ - m_1{}^\circ}{n^\circ} < 1$,

but it also equals the integer $\mu^\circ - \omega$; this integer then must equal zero, or $\omega = \mu^\circ$ and $m^\circ = m_1{}^\circ$. In the same manner we can shew that the quotient which precedes ω is equal to that which precedes μ°, and so on to the quotient a, so that a is the quotient which first recurs and with which therefore the period commences.

Hence the quotients and converging fractions may now be represented by

$$a;\ a, \beta, \ldots \lambda, \quad \mu;\ a, \beta, \ldots \lambda, \quad \mu;\ a, \beta, \ldots .$$

$$\frac{a}{1}, \ldots \frac{p^\circ}{q^\circ}, \frac{p}{q}; \frac{p'}{q'}, \ldots \frac{p_1{}^\circ}{q_1{}^\circ}, \frac{p_1}{q_1}; \frac{p_1'}{q_1'}, \ldots$$

Let z be the complete quotient of which μ, the last partial quotient in the first period, is the integral part, then $z - \mu = \sqrt{\mathrm{N}} - a$;

$$\therefore\ \sqrt{\mathrm{N}} = \frac{pz + p^\circ}{qz + q^\circ} = \frac{p\sqrt{\mathrm{N}} + p(\mu - a) + p^\circ}{q\sqrt{\mathrm{N}} + q(\mu - a) + q^\circ};$$

$$\therefore\ p(\mu - a) + p^{\circ} = Nq, \quad q(\mu - a) + q^{\circ} = p;$$

$\therefore\ \mu - a + \dfrac{q^{\circ}}{q} = \dfrac{p}{q}$, or $\mu - a$ is the greatest integer in $\dfrac{p}{q}$ and therefore $= a$, $\therefore\ \mu = 2a$.

Lastly, since $q^{\circ} = p - aq$, and $\dfrac{p^{\circ} - aq^{\circ}}{q^{\circ}}$, $\dfrac{p - aq}{q}$, are consecutive fractions converging to the value of $\sqrt{N} - a$ which equals $\dfrac{1}{a+}\,\dfrac{1}{\beta+}\ \ldots\ \dfrac{1}{\lambda}$, therefore (Art. 145) the period of quotients $a, \beta, \gamma \ldots . \kappa, \lambda$ is the same taken from the beginning and end, *i. e.* $\lambda = a$, $\kappa = \beta, \ldots .$ Hence the quotients proceed according to the law

$$a;\ a, \beta, \gamma, \ldots \gamma, \beta, a, 2a;\ a, \beta, \ldots \beta, a, 2a;\ a \ldots$$

which law would be yet more regular if the first quotient were $2a$ or zero; *i. e.* if the irrational quantity developed were $\sqrt{N} \pm a$ instead of $\sqrt{N}$.

147. Every converging fraction $\dfrac{p}{q}$ which corresponds to the quotient $2a$ in any period, is such that $p^2 - Nq^2 = \pm 1$. For since $\mu = 2a$, the equation $m + m' = \mu n$, in which neither m nor m' can exceed a, will necessarily give $m = m' = a$, and $n = 1$; therefore the equation $(pq^{\circ} - qp^{\circ})\,n = p^2 - Nq^2$ becomes $p^2 - Nq^2 = \pm 1$ according as $\dfrac{p}{q} >$ or $< \sqrt{N}$.

Hence the equation $x^2 - Ny^2 = \pm 1$ may be always solved in whole numbers (at least with the upper sign) whatever be the number N (provided it be not a perfect square), in an infinite number of ways. If the number of terms in the period $a, \beta, \ldots . \beta, a, 2a$, be even, all the fractions in the different periods corresponding to $2a$ will be $> \sqrt{N}$, and we shall obtain solutions only of $x^2 - Ny^2 = +1$; but if the

period consist of an odd number of terms, then the first fraction which corresponds to $2a$ will be $< \sqrt{N}$, the second fraction corresponding to $2a > \sqrt{N}$, and so on; so that all fractions corresponding to $2a$ which stand in odd places will satisfy $x^2 - Ny^2 = -1$, and those in even places the equation $x^2 - Ny^2 = +1$.

Ex. 1. $$x^2 - 23y^2 = 1.$$

For $\sqrt{23}$ we have (p. 133) the quotients and converging fractions,

$$4,\ 1,\ 3,\ 1,\ 8,\quad 1,\quad 3,\quad 1,\quad 8, \ldots$$

$$\frac{4}{1}, \frac{5}{1}\ \frac{19}{4}\ \frac{24}{5}, \ldots\ldots\ldots\ldots \frac{1151}{240}, \ldots$$

$\therefore x = 24, y = 5$; or $x = 1151, y = 240$; ....

Ex. 2. $$x^2 - 13y^2 = \mp 1.$$

With the upper sign $x = 18, \quad y = 5$;

with the lower $\qquad x = 649, \ y = 180$.

The above investigation of the properties of the continued fraction which expresses $\sqrt{N}$, is taken from Legendre's *Essai sur la Théorie des Nombres*.

SECTION VIII.

ON THE SYMMETRICAL FUNCTIONS OF THE ROOTS OF AN EQUATION.

148. A symmetrical function of the roots of an equation, as was before observed, is an expression in which each root is alike involved; and which is consequently made up of all the roots in such a manner that if any two be interchanged, its value is not altered. Thus

$$-p_1 = a + b + c + \ldots + l, \; p_2 = ab + ac + bc + \ldots,$$

and, in general, all the coefficients are symmetrical functions of the roots; for in these expressions, if b were written in every place where a occurs instead of a, and a in every place where b occurs instead of b, or if any other two of the roots were interchanged, the values of the expressions would not be altered.

149. We shall first consider the elementary cases where, in each term, only one, two, three, &c. of the roots are involved; viz.

$$a^m + b^m + c^m + d^m + \ldots .$$

$$a^m b^p + a^m c^p + b^m c^p + \ldots .$$

$$a^m b^p c^q + a^m c^p b^q + b^m a^p c^q + \ldots .$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

The first is formed by taking the sum of the roots each raised to the same power m, and consists of n terms.

The second is formed by taking all the permutations of the roots taken two together, and affecting the first letter in each product with the index m, and the second with the index p; and it consists of $n(n-1)$ terms.

The third is formed by taking all the permutations of the roots taken three together, and affecting the first letter in each product with the index m, the second with the index p, and the third with the index q; and it consists of $n(n-1)(n-2)$ terms.

Similarly, the symmetrical function each term of which contained r roots, would be formed by taking all the permutations of the roots taken r together, and in each product affecting the first letter with the index m, the second with the index p, the third with the index q, and so on; and it would consist of $n(n-1)(n-2)\ldots(n-r+1)$ terms. (The above supposes all the indices $m, p, q, \ldots.$ to be unequal; we shall afterwards return to the case where some of them are equal).

Since therefore in the above cases, any term being given, all the others may be deduced from it, by forming all the permutations of the letters which compose it and affecting the letters in each with the indices taken always in the same order; we may denote them by the symbol Σ prefixed to any one of the terms, thus

$$\Sigma(a^m), \quad \Sigma(a^m b^p), \quad \Sigma(a^m b^p c^q);$$

and the first, that is, the sum of the m^{th} powers of the roots may be indifferently expressed by $\Sigma(a^m)$ or S_m; we shall generally employ the latter, as the sums of similar powers of the roots are the simplest sort of symmetrical functions, and are the quantities by which all others are expressed.

150. The value of every *rational* symmetrical function of

the roots of an equation can be expressed by the coefficients, without knowing the actual values of the roots, as we shall shew. But it will be necessary to consider only the case of integral functions; because when the terms of a symmetrical function are fractional, we can, by reducing them to a common denominator, express the function by a single fraction whose numerator and denominator are integral symmetrical functions.

Thus $\dfrac{ab}{2c^2} + \dfrac{ac}{2b^2} + \dfrac{bc}{2a^2} - 3abc$, which is a fractional symmetrical function of the three quantities a, b, c, becomes by reduction $\dfrac{a^2b^2 + a^2c^2 + b^2c^2 - 6a^3b^3c^3}{2a^2b^2c^2}$.

151. To express the sum of the m^{th} powers of the roots of an equation in terms of the coefficients, and the sums of the inferior powers.

We have (Art. 60)

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \ldots\ldots + \frac{f(x)}{x-l};$$

therefore, effecting the divisions, which can all be exactly performed since $a, b, \ldots\ldots l$ are roots of $f(x) = 0$, we have (Art. 6)

$$\frac{f(x)}{x-a} = x^{n-1} + (a + p_1)\, x^{n-2} + (a^2 + p_1 a + p_2)\, x^{n-3} + \ldots$$
$$+ (a^m + p_1 a^{m-1} + p_2 a^{m-2} + \ldots\ldots + p_m)\, x^{n-m-1} + \ldots\ldots$$

$$\frac{f(x)}{x-b} = x^{n-1} + (b + p_1)\, x^{n-2} + (b^2 + p_1 b + p_2)\, x^{n-3} + \ldots\ldots$$
$$+ (b^m + p_1 b^{m-1} + p_2 b^{m-2} + \ldots\ldots + p_m)\, x^{n-m-1} + \ldots\ldots$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{f(x)}{x-l} = x^{n-1} + (l + p_1)\, x^{n-2} + (l^2 + p_1 l + p_2)\, x^{n-3} + \ldots\ldots$$
$$+ (l^m + p_1 l^{m-1} + p_2 l^{m-2} + \ldots\ldots + p_m)\, x^{n-m-1} + \ldots\ldots$$

Hence, adding these quotients together, we have

$$f'(x) = nx^{n-1} + (S_1 + np_1)\,x^{n-2} + (S_2 + p_1 S_1 + np_2)\,x^{n-3} + \ldots$$
$$+ (S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots + p_{m-1} S_1 + np_m)\,x^{n-m-1} + \ldots .$$

But $f'(x) = nx^{n-1} + (n-1)\,p_1 x^{n-2} + (n-2)\,p_2 x^{n-3} + \ldots$
$$\ldots + (n-m)\,p_m x^{n-m-1} + \ldots ,$$

hence, equating the coefficients of corresponding terms in these identical expressions, we have

$$S_1 + np_1 = (n-1)\,p_1, \text{ or } S_1 + p_1 = 0,$$
$$S_2 + p_1 S_1 + np_2 = (n-2)\,p_2, \text{ or } S_2 + p_1 S_1 + 2p_2 = 0, \ldots$$
$$S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots p_{m-1} S_1 + np_m = (n-m)\,p_m,$$
$$\text{or } S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots + p_{m-1} S_1 + mp_m = 0,$$

the formula which gives the sum of the m^{th} powers (m being less than n) of the roots, in terms of the coefficients and the sums of the inferior powers; and by means of which the sums of all similar powers whose index is less than the degree of the equation, can be successively expressed by integral functions of the coefficients.

But if m be equal to or greater than n, multiplying the equation by x^{m-n}, we have

$$x^m + p_1 x^{m-1} + p_2 x^{m-2} + \ldots + p_n x^{m-n} = 0;$$

therefore, replacing x successively by all the roots $a, b, c, \ldots l$, and taking the sum of the results, we have

$$S_m + p_1 S_{m-1} + p_2 S_{m-2} + \ldots + p_n S_{m-n} = 0.$$

Hence making $m = n,\ n+1,\ n+2, \ldots$ successively, we find, observing that

$$S_0 = a^\circ + b^\circ + c^\circ + \ldots + l^\circ = n,$$
$$S_n + p_1 S_{n-1} + p_2 S_{n-2} + \ldots + np_n = 0,$$
$$S_{n+1} + p_1 S_n + p_2 S_{n-1} + \ldots + p_n S_1 = 0, \ldots$$

Hence the sums of all similar powers whatever of the roots can be expressed by integral functions of the coefficients.

152. To find the sums of the negative similar powers of the roots, we must write $\frac{1}{y}$ for x, and apply the above formulæ to the transformed equation in y.

153. We may observe that by the preceding method the value of $\phi(a) + \phi(b) + \dots + \phi(l)$, (where $\phi(x)$ denotes any rational algebraic function) may be readily found. For

$$\frac{f'(x) \cdot \phi(x)}{f(x)} = \frac{\phi(x)}{x-a} + \frac{\phi(x)}{x-b} + \dots \frac{\phi(x)}{x-l},$$

therefore, performing the divisions, and reserving only the remainders, (Art. 6)

$$\frac{Ax^{n-1} + Bx^{n-2} + \dots}{f(x)} = \frac{\phi(a)}{x-a} + \frac{\phi(b)}{x-b} + \dots + \frac{\phi(l)}{x-l};$$

$$\therefore Ax^{n-1} + Bx^{n-2} + \dots = x^{n-1}\{\phi(a) + \phi(b) + \dots \phi(l)\} + \dots$$

$\therefore \phi(a) + \phi(b) + \dots + \phi(l) = A =$ coefficient of the highest power of x, in the remainder of the division $f'(x).\phi(x)$ by $f(x)$.

154. In practical applications to equations of a low degree, or consisting of a small number of terms, we may, instead of calculating the sums of the powers successively from one another, express them immediately in terms of the coefficients of the equation, by the following method.

For x write $\frac{1}{y}$ in the identical equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \dots + p_n = (x-a)(x-b)(x-c)\dots(x-l);$$

$$\therefore 1 + p_1y + p_2y^2 + p_3y^3 + \dots + p_ny^n = (1-ay)(1-by)(1-cy)\dots(1-ly).$$

Hence, taking the *Napierian* logarithms of both sides,

$$\left.\begin{matrix} p_1 y + p_2 \\ -\frac{1}{2}p_1^2 \end{matrix}\right\} y^2 + \left.\begin{matrix} p_3 \\ -p_1 p_2 \\ +\frac{1}{3}p_1^3 \end{matrix}\right\} y^3 + \left.\begin{matrix} p_4 \\ -p_1 p_3 \\ -\frac{1}{2}p_2^2 \\ +p_1^2 p_2 \\ -\frac{1}{4}p_1^4 \end{matrix}\right\} y^4 + \dots$$

$$= -yS_1 - \tfrac{1}{2}y^2S_2 - \tfrac{1}{3}y^3S_3 - \tfrac{1}{4}y^4S_4 - \dots .$$

therefore, equating coefficients,

$$S_1 = -p_1$$
$$S_2 = -2p_2 + p_1^2$$
$$S_3 = -3p_3 + 3p_1p_2 - p_1^3$$
$$S_4 = -4p_4 + 4p_1p_3 + 2p_2^2 - 4p_1^2p_2 + p_1^4, \text{ \&c.}$$

Ex. 1. $\qquad x^4 + rx + s = 0.$

Let $x^4 + rx + s = (x-a)(x-b)(x-c)(x-d)$;

$\therefore\ 1 + y^3(r+sy) = (1-ay)(1-by)(1-cy)(1-dy)$;

$$\therefore y^3(r+sy) - \tfrac{1}{2}y^6(r+sy)^2 + \dots = -yS_1 - \tfrac{1}{2}y^2S_2 - \tfrac{1}{3}y^3S_3$$
$$- \tfrac{1}{4}y^4S_4 - \tfrac{1}{5}y^5S_5 - \tfrac{1}{6}y^6S_6 - \dots .$$

hence, equating coefficients, we have

$S_1 = 0,\ S_2 = 0,\ S_3 = -3r,\ S_4 = -4s,\ S_5 = 0,\ S_6 = 3r^2.$

Ex. 2. The sum of the m^{th} powers of the roots of $x^n - 1 = 0$ is n, when m is a multiple of n; and zero in all other cases.

Let $\quad x^n - 1 = (x-a)(x-b)\dots(x-l)$;

$$\therefore\ 1 - y^n = (1-ay)(1-by)\dots(1-ly);$$

$$\therefore y^n + \tfrac{1}{2}y^{2n} + \dots + \frac{1}{r}y^{rn} + \dots = yS_1 + \tfrac{1}{2}y^2S_2 + \dots \frac{1}{rn}y^{rn}S_{rn} + \dots$$

$$\therefore\ S_1 = S_2 = \dots = 0,$$
$$S_n = S_{2n} = \dots = S_{rn} = n.$$

Ex. 3. To express the sum of the m^{th} powers of the roots of a quadratic in terms of its coefficients.

Let $$x^2 - px + q = (x-a)(x-b);$$

$$\therefore 1 - y(p - qy) = (1-ay)(1-by);$$

therefore, taking the *Napierian* logarithms of both sides, and writing down only the terms which, when developed, will involve y^m, we have

$$\frac{y^m}{m}(p-qy)^m + \frac{y^{m-1}}{m-1}(p-qy)^{m-1} + \frac{y^{m-2}}{m-2}(p-qy)^{m-2} + \ldots$$
$$= \frac{1}{m} y^m S_m + \ldots.$$

therefore, equating coefficients of y^m,

$$S_m = p^m - mp^{m-2}q + \frac{m(m-3)}{1.2} p^{m-4}q^2 - \ldots.$$
$$+ (-1)^r \frac{m(m-r-1)(m-r-2)..(m-2r+1)}{1.2.3\ldots.r} p^{m-2r}q^{2r} + \ldots$$

155. Similarly, we may express the coefficients immediately in terms of the sums of the powers. For since (Art. 154)

$$\log_e(1+p_1y+p_2y^2+p_3y^3+\ldots+p_ny^n) = -yS_1 - \tfrac{1}{2}y^2S_2 - \tfrac{1}{3}y^3S_3 - \ldots.$$

$$\therefore 1+p_1y+p_2y^2+p_3y^3+\ldots = e^{-yS_1-\frac{1}{2}y^2S_2-\frac{1}{3}y^3S_3-\ldots}$$
$$= 1 - yS_1 \left.\begin{matrix} -\tfrac{1}{2}S_2 \\ +\tfrac{1}{2}(S_1)^2 \end{matrix}\right\} y^2 \left.\begin{matrix} -\tfrac{1}{3}S_3 \\ +\tfrac{1}{2}S_1S_2 \\ -\tfrac{1}{6}(S_1)^3 \end{matrix}\right\} y^3 - \ldots$$

hence, equating coefficients,

$$p_1 = -S_1$$
$$p_2 = -\tfrac{1}{2}S_2 + \frac{1}{1.2}(S_1)^2$$
$$p_3 = -\tfrac{1}{3}S_3 + \frac{1}{1.2}S_1S_2 - \frac{1}{1.2.3}(S_1)^3$$
$$\ldots = \ldots.$$

Ex. $x^6 + p_2x^4 + p_3x^3 + p_4x^2 + p_5x + p_6 = 0.$

Here $S_1 = 0$, and proceeding as above, and in the development of the second member reserving only powers of y as far as the sixth, we have

$$1 + p_2y^2 + p_3y^3 + p_4y^4 + p_5y^5 + p_6y^6 = e^{-\frac{1}{2}y^2S_2-\frac{1}{3}y^3S_3- \cdots}$$

$$= 1 - \frac{y^2}{1}(\tfrac{1}{2}S_2 + \tfrac{1}{3}yS_3 + \tfrac{1}{4}y^2S_4 + \tfrac{1}{5}y^3S_5 + \tfrac{1}{6}y^4S_6)$$

$$+\frac{y^4}{1.2}(\tfrac{1}{2}S_2 + \tfrac{1}{3}yS_3 + \tfrac{1}{4}y^2S_4)^2 - \frac{y^6}{1.2.3}(\tfrac{1}{2}S_2)^3;$$

$$\therefore p_2 = -\tfrac{1}{2}S_2,\ p_3 = -\tfrac{1}{3}S_3,$$

$$p_4 = -\tfrac{1}{4}S_4 + \frac{1}{1.2}(\tfrac{1}{2}S_2)^2$$

$$p_5 = -\tfrac{1}{5}S_5 + \frac{1}{1.2}\tfrac{1}{3}S_2S_3$$

$$p_6 = -\tfrac{1}{6}S_6 + \frac{1}{1.2}\left(\tfrac{1}{4}S_2S_4 + \frac{1}{9}(S_3)^2\right) - \frac{1}{1.2.3}(\tfrac{1}{2}S_2)^3.$$

156. The employment of the sums of similar powers of the roots, was first given by *Newton* as a method of approximating to the greatest root.

Suppose $a, b, c \ldots.$ to be the roots arranged in order of magnitude, then

$$\frac{S_{m+1}}{S_m} = \frac{a^{m+1} + b^{m+1} + \ldots}{a^m + b^m + \ldots} = a.\ \frac{1 + \left(\frac{b}{a}\right)^{m+1} + \ldots}{1 + \left(\frac{b}{a}\right)^m + \ldots}$$

which (since the fractions $\left(\frac{b}{a}\right)^m$, $\left(\frac{b}{a}\right)^{m+1}$, ... may, by the increase of m, be made as small as ever we please) approaches to a as its limit; and therefore $\frac{S_{m+1}}{S_m}$ is an approximation to the greatest root, becoming closer and closer as m increases.

There is however one case in which the method fails, viz. when the equation has a pair of imaginary roots ρ (cos $\theta \pm \sqrt{-1}$ sin θ), whose modulus ρ exceeds the greatest of the real roots; for the sum of the m^{th} powers of these two roots is $2\rho^m \cos m\theta$, and a^m cannot in all cases be made to surpass this, unless a be greater than ρ.

157. The theory of symmetrical functions will enable us to transform an equation, whose roots are unknown, into another whose roots are all the combinations, formed after an assigned law, of the roots of the proposed, taken two, three, &c. at a time. We shall exemplify the method in the following transformation, as being the most convenient practical one of solving a problem of considerable interest.

158. To transform an equation into one whose roots are the squares of the differences of its roots.

Let $a, b, c, \ldots l$ be the n roots of the proposed, then the roots of the transformed equation will be

$$(a-b)^2, \ (a-c)^2, \ (b-c)^2 \ldots$$

in number $\frac{1}{2}n\,(n-1)$, since they include all the combinations of the n quantities $a, b, c, \ldots . l$ taken two together; hence the degree of the transformed equation will be $\frac{1}{2}n\,(n-1) = m$ suppose.

Let the transformed equation be

$$y^m + q_1 y^{m-1} + q_2 y^{m-2} + q_3 y^{m-3} + \ldots + q_m = 0,$$

and let $s_1, s_2, \ldots . s_i$ denote the sums of the first, second, and i^{th} powers of its roots; then (Art. 151) all the coefficients may be expressed by these sums, thus

$$q_1 = -s_1, \ q_2 = -\tfrac{1}{2}(s_2 + q_1 s_1), \ q_3 = -\tfrac{1}{3}(q_2 s_1 + q_1 s_2 + s_3), \ldots$$

therefore it only remains to calculate $s_1, s_2, \ldots$ Now if $S_1, S_2 \ldots$

denote, as usual, the sums of the powers of the roots of the proposed equations, and k be any positive integer, we have

$$(x-a)^k+(x-b)^k+\ldots+(x-l)^k=nx^k-kS_1x^{k-1}$$
$$+\frac{k(k-1)}{1.2}S_2x^{k-2}-\ldots+(-1)^kS_k.$$

Therefore, changing x successively into $a, b, c \ldots . l$, and taking the sum of the resulting equations, we have

$$(a-b)^k+(a-c)^k+\ldots+(a-l)^k+(b-a)^k+(b-c)^k+\ldots$$
$$+(b-l)^k+\ldots .$$
$$=nS_k-kS_1S_{k-1}+\frac{k(k-1)}{1.2}S_2S_{k-2}-\ldots+(-1)^knS_k.$$

Now if k be an odd number, each member of this equation is separately zero; but if k be an even number and $=2i$, then the value of the first member is $2s_i$; and in the second member, the terms are equal taken from the beginning and end;

$$\therefore s_i=nS_{2i}-2iS_1S_{2i-1}+\frac{2i(2i-1)}{1.2}S_2S_{2i-2}-\ldots$$
$$+\tfrac{1}{2}(-1)^i\frac{2i(2i-1)\ldots(i+1)}{1.2.3\ldots i}S^2_i.$$

Hence to form the equation whose roots are the squares of the differences of the roots of $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, or, as it is called, the equation of differences, we must first calculate $S_1, S_2, S_3, \ldots$ in terms of $p_1, p_2 \ldots$; and next $s_1, s_2, s_3, \ldots$ by the formula just investigated; and lastly $q_1, q_2, q_3, \ldots$ the coefficients of the required equation, by the method of (Art. 151).

159. We have seen one use of the equation of differences (Art. 49), viz. to determine a limit less than the least difference of the roots of a proposed equation; another is to determine, within certain limits, the number of impossible roots which the proposed equation contains. If the transformed equation

be complete and have no continuations of sign, it cannot have a negative root; and therefore the primitive equation has no impossible roots, because a pair must give rise to a real negative root in the equation of differences; but if the transformed equation have continuations of sign, then it has either impossible or negative roots; and as these can only arise from impossible roots in the proposed equation, it follows that this latter has impossible roots. Also if the proposed equation has p possible roots, the transformed equation will have $\frac{p(p-1)}{2}$ positive roots, and the rest will be either negative or imaginary; hence if the last term of the transformed equation be positive, $\frac{p(p-1)}{2}$ is even; and therefore p, which must be of the same parity as n, will be of the form $4m$ or $4m+1$, according as n is even or odd. Similarly, if the last term of the transformed equation be negative, it may be shewn that the number of real roots in the proposed equation will be of the form $4m+2$ or $4m+3$, according as n is even or odd.

Ex. To transform $x^4 + rx + s = 0$ into one whose roots shall be the squares of the differences of its roots.

Here, by Ex. (Art. 154), and by the formulæ of Arts. 158 and 151, we have

$$S_1 = 0, \quad S_2 = 0, \quad S_3 = -3r, \quad S_4 = -4s, \quad S_5 = 0,$$
$$S_6 = 3r^2, \quad S_7 = 7rs, \quad S_8 = 4s^2, \quad S_9 = -3r^3,$$
$$S_{10} = -10r^2 s, \quad S_{11} = -11rs^2, \quad S_{12} = -4s^3 + 3r^4;$$
$$s_1 = 0, \quad s_2 = -16s, \quad s_3 = -78r^2, \quad s_4 = 576s^2,$$
$$s_5 = -40r^2 s, \quad s_6 = -7936s^3 + 2190r^4;$$
$$q_1 = 0, \quad q_2 = 8s, \quad q_3 = 26r^2, \quad q_4 = -112s^2,$$
$$q_5 = 216r^2 s, \quad q_6 = 256s^3 - 27r^4.$$

Hence the transformed equation is

$$y^6 + 8sy^4 + 26r^2 y^3 - 112s^2 y^2 + 216r^2 sy + 256s^3 - 27r^4 = 0.$$

Hence if the last term be positive, or $\left(\frac{s}{3}\right)^3 > \left(\frac{r}{4}\right)^4$, the number of real roots in the proposed will be of the form of $4m$; but there cannot be more than two, therefore there are none. If the last term be negative, or $\left(\frac{s}{3}\right)^3 < \left(\frac{r}{4}\right)^4$, the number of real roots of the proposed will be of the form $4m + 2$, and therefore there will be two. These results agree with those found at p. 54.

160. Every rational symmetrical function of the roots of an equation can be expressed by the coefficients of that equation.

First, to find the value of the double function $\Sigma(a^m b^p)$.

If we multiply together the two equations

$$S_m = a^m + b^m + c^m + \ldots\ldots + l^m$$

$$S_p = a^p + b^p + c^p + \ldots\ldots + l^p,$$

we have

$$S_m S_p = a^{m+p} + b^{m+p} + c^{m+p} + \ldots\ldots + l^{m+p}$$
$$+ a^m b^p + a^m c^p + b^m a^p + \ldots\ldots$$

Now the first line is equal to S_{m+p}; and the second consists of all the permutations of the roots taken two together, the first letter in each being affected with the index m, and the second with the index p, and is therefore equal to the double function $\Sigma(a^m b^p)$;

$$\therefore S_m S_p = S_{m+p} + \Sigma(a^m b^p),$$

$$\text{or } \Sigma(a^m b^p) = S_m S_p - S_{m+p} \ldots (1).$$

Next to find the value of the triple function $\Sigma(a^m b^p c^q)$.

Multiplying together the equations

$$\Sigma(a^m b^p) = a^m b^p + a^m c^p + b^m a^p + \ldots .$$

$$S_q = \quad a^q + \quad b^q + c^q \quad + \ldots .$$

the result will consist of three different partial products,

(1) the sum of products of the form $a^{m+q}b^p = \Sigma(a^{m+q}b^p)$,

(2) the sum of products of the form $a^m b^{p+q} = \Sigma(a^m b^{p+q})$,

(3) the sum of products of the form $a^m b^p c^q = \Sigma(a^m b^p c^q)$;

$$\therefore \Sigma(a^m b^p) S_q = \Sigma(a^{m+q}b^p) + \Sigma(a^m b^{p+q}) + \Sigma(a^m b^p c^q).$$

Hence, replacing $\Sigma(a^m b^p)$, $\Sigma(a^{m+q}b^p)$, $\Sigma(a^m b^{p+q})$, by their values obtained from formula (1), we have

$$\Sigma(a^m b^p c^q) = S_m S_p S_q - S_{m+p} S_q - S_{m+q} S_p - S_{p+q} S_m + 2S_{m+p+q}.$$

In the same manner might the quadruple function $\Sigma(a^m b^p c^q d^r)$, or the sum of any succeeding combinations, be expressed by the sums of the powers; and as these latter are expressible by integral functions of the coefficients, it follows that all the above symmetrical functions can be expressed by integral functions of the coefficients. And as every symmetrical polynomial in $a, b, c \ldots$ must be composed of the assemblage, by addition or subtraction, of several symmetrical functions of the form $\Sigma(a^m b^p c^q \ldots)$, it follows that the value of every rational symmetrical function whatever of the roots of an equation (without the roots being known) can be expressed by the coefficients of the equation.

161. The above expressions for the elementary symmetrical functions will require to be modified when any of the indices become equal. Thus if $m = p$ in the formula

$$\Sigma(a^m b^p) = S_m S_p - S_{m+p},$$

since $a^m b^p = b^m a^p$, the terms in $\Sigma(a^m b^p)$ will become equal two and two, and $\Sigma(a^m b^p)$ will be reduced to $2\Sigma(a^m b^m)$;

$$\therefore \Sigma(a^m b^m) = \tfrac{1}{2}(S_m^2 - S_{2m}).$$

Similarly, if $m = p = q$ in $\Sigma(a^m b^p c^q)$, the six combinations formed by interchanging a, b, c in $a^p b^q c^r$ are reduced to one, and $\Sigma(a^m b^p c^q)$ is reduced to $6\Sigma(a^m b^m c^m)$;

$$\therefore\ \Sigma(a^m b^m c^m) = \tfrac{1}{6}S_m{}^3 - \tfrac{1}{2}S_{2m}S_m + \tfrac{1}{3}S_{3m};$$

and in general, if t of the exponents become equal, the general formula must be divided by $1.2.3 \ldots . t$. If $\Sigma(a^m b^p c^q \ldots .)$ have r roots in each term, it will consist, as we have seen, of $n(n-1) \ldots . (n-r+1)$ terms; and if t of the indices become equal, it will consist of $\dfrac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots t}$ terms.

Ex. 1. Let the roots of $x^3 + px^2 + qx + r = 0$ be a, b, c, to find the value of $\Sigma(a^2 b)$.

$$\begin{aligned}\Sigma(a^2 b) &= S_1 S_2 - S_3 \\ &= (-p)(p^2 - 2q) - (-3r + 3pq - p^3) \text{ (Art. 154)} \\ &= 3r - pq.\end{aligned}$$

Ex. 2. Let the roots of $x^n - 1 = 0$ be a, b, c, &c., to find the value of $\Sigma(a^m b^p)$.

$$\Sigma(a^m b^p) = S_m S_p - S_{m+p}.$$

Hence the value is $n^2 - n$, when m and p are both multiples of n; and $-n$, when $m + p$ only is a multiple of n; and zero, in all other cases, (Ex. 2, Art. 154).

162. Besides the method of Art. 158, we may also transform an equation by means of symmetrical functions, as follows.

Suppose that each root of the transformed equation is to be a rational function $\phi(a, b, c, \ldots)$ of any number of the roots of the proposed equation; then having formed all the combinations $\phi(a, b, c, \ldots)$, $\phi(a, c, d, \ldots)$, &c., the transformed equation, resolved into its factors, will be

$$\{y - \phi(a, b, c, \ldots)\}\ \{y - \phi(a, c, d, \ldots)\} \ldots = 0;$$

and as this product is not altered by interchanging $a, b, c, \ldots$ among themselves, (for the only effect of that is to place its factors in a different order,) we are certain that, after multiplication, the coefficients of the different powers of y will be symmetrical functions of $a, b, c, \ldots$, and may therefore be expressed by the coefficients of the proposed equation.

Ex. To transform $x^4 + px^3 + qx^2 + rx + s = 0$, roots a, b, c, d, into one whose roots shall be

$$ab + cd, \quad ac + bd, \quad ad + bc;$$

the transformed equation is

$$\{y - (ab + cd)\}.\{y - (ac + bd)\}.\{y - (ad + bc)\} = 0, \text{ or}$$

$$y^3 - qy^2 + (pr - 4s)\,y - (r^2 - 4qs + p^2s) = 0, \ldots (1).$$

Hence also we can find an equation whose roots are

$$(a + b - c - d)^2, \quad (a + c - b - d)^2, \quad (a + d - b - c)^2.$$

For let $z = (a + b - c - d)^2$

$$= (a + b + c + d)^2 - 4(ab + ac + ad + bc + bd + cd)$$
$$+ 4(ab + cd) = p^2 - 4q + 4y;$$

$$\therefore y = \frac{z - p^2 + 4q}{4};$$

and substituting in (1), the transformed equation in z is

$$z^3 - (3p^2 - 8q)\,z^2 + (3p^4 - 16p^2q + 16q^2 + 16pr - 64s)z$$
$$- (p^3 - 4pq + 8r)^2 = 0.$$

Either of these transformed equations may be employed in the solution of the proposed biquadratic. Thus let α be a value of y, then $ab + cd = \alpha$; $abcd = s$; therefore ab and cd are known; also $ab(c + d) + cd(a + b) = -r$, $(a + b) + (c + d) = -p$; therefore $a + b$, and $c + d$ are known; hence all the roots are obtained from one root of the reducing cubic. In the second case, if we know the three values of z, by means of these, and the equation $a+b+c+d = -p$, we can find the roots of the biquadratic merely by addition and subtraction.

163. Every equation of an even degree has at least one real quadratic factor.

Let the proposed equation be

$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$, having roots $a, b, c, \ldots$;

and let $n = 2\mu$, μ being an odd number. Let it be transformed (Art. 162) into an equation whose roots are the combinations of every two of its roots, of the form $y = a + b + mab$, m being any number; also let the transformed equation be $\phi_m(y) = 0$; then its coefficients will be symmetrical functions of $a, b, c, \ldots$ and therefore rational and known functions of $p_1, p_2, \ldots$ and its degree will be $\dfrac{2\mu(2\mu - 1)}{2}$ which is odd; therefore $\phi_m(y) = 0$ will have at least one real root, whatever be the value of m. Hence making $m = 1, 2, 3, \ldots \mu(2\mu - 1) + 1$, successively, each of the equations $\phi_1(y) = 0$, $\phi_2(y) = 0, \ldots$ will have at least one real root; that is, we shall have $\mu(2\mu - 1) + 1$ real values for combinations of two roots of the proposed equation, of the form $a + b + mab$; but there are only $\mu(2\mu - 1)$ such combinations which are differently composed of the roots $a, b, c, \ldots$; therefore two of these combinations for which we have obtained real values, must involve the same pair of the quantities $a, b, c, \ldots$; let this pair of roots be a, b, and α, α' the real roots of the corresponding equations $\phi_m(y) = 0$, $\phi_{m'}(y) = 0$, so that

$$a + b + mab = \alpha, \quad a + b + m'ab = \alpha';$$

therefore $a + b$ and ab are real, and the proposed equation has at least one real quadratic factor, and two roots, either real, or of the form $r(\cos\theta \pm \sqrt{-1}\sin\theta)$. Hence every equation whose degree is only once divisible by 2, may be depressed two dimensions, so that the reduced equation may still have rational coefficients.

We shall now prove that if it be true that every equation has at least one real quadratic factor when its degree is r times divisible by 2, or when $n = 2^r\mu$ where μ is odd, the same is true when the degree of the equation is $r + 1$ times divisible by 2. For let $n = 2^{r+1}\mu$; then the degree of the transformed equation will be $2^r\mu(2^{r+1}\mu - 1)$ which is only r times divisible by 2; therefore the transformed equation $\phi_m(y) = 0$ will have two roots either real, or of the form $y = \alpha \pm \beta\sqrt{-1}$, each of which expresses the value of some one of the combinations $a + b + mab$, $a + c + mac$, ... Suppose that we have $a + b + mab = \alpha + \beta\sqrt{-1}$, β being zero if the value of y be real; then, as shewn above, we can give m such a value m', that $\phi_{m'}(y) = 0$ shall have a root corresponding to the combination of the same letters, so that $a + b + m'ab = \alpha' + \beta'\sqrt{-1}$; from which equations we can obtain values of ab and $a + b$ under the forms

$$a + b = r(\cos\theta + \sqrt{-1}\sin\theta)$$

$$ab = r'(\cos\theta' + \sqrt{-1}\sin\theta'),$$

and then we may successively obtain $a - b$ and a and b, and reduce the values to the forms

$$a - b = \rho(\cos\phi + \sqrt{-1}\sin\phi)$$

$$a = \mathrm{R}(\cos\psi + \sqrt{-1}\sin\psi),\ b = \mathrm{R}(\cos\psi - \sqrt{-1}\sin\psi).$$

Hence the proposed will have a real quadratic factor, provided an equation whose degree $= 2^r\mu$, has one; but we have proved this to be the case when $r = 1$; therefore it is universally true that every equation of an even degree has at least one real quadratic factor. If now this factor be expelled, the depressed equation will have its coefficients rational and its degree even, and will therefore, as before, have one real quadratic factor. Hence the first member of every equation of an even degree may be resolved into real quadratic factors.

164. Hence if we divide the first member of any equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_n = 0$$

by $x^2 + ax + b$, admitting no terms into the quotient that have x in the denominator, we shall at last obtain a remainder of the form $Ax + B$, A and B being rational functions of a and b; and in order that $x^2 + ax + b$ may be a quadratic factor of the proposed equation, it is necessary and sufficient that this remainder should equal zero for all values of x, which requires that we separately have $A = 0$, $B = 0$. The different pairs of values real or imaginary of a and b which satisfy these equations, will give all the quadratic factors of the proposed; and as the number of these factors is $\frac{1}{2}n(n-1)$ (Art. 17), the final equation for determining one of the quantities a, b, obtained by eliminating the other between the two preceding equations, will be of the degree $\frac{1}{2}n(n-1)$, which exceeds n if $n > 3$; therefore the determination of the quadratic factors of an equation will generally present greater difficulties than the solution of the equation. As the proposed equation has necessarily $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$ real quadratic factors, according as n is even or odd, there will always exist the same number of pairs of real values of a and b, satisfying the equations $A = 0$, $B = 0$; and if any of these pairs of real values be commensurable, they may be easily found; and the commensurable quadratic factors being known, the equation may be depressed.

Ex. 1. The resolution of $x^4 + px^3 + qx^2 + rx + s$ into its two quadratic factors $x^2 + mx + n$, $x^2 + m'x + n'$, may be effected by the following formulæ:

$$m = \tfrac{1}{2}(p + \sqrt{z}), \qquad m' = \tfrac{1}{2}(p - \sqrt{z}),$$

$$n = \frac{r - qm + pm^2 - m^3}{p - 2m}, \quad n' = \frac{r - qm' + pm'^2 - m'^3}{p - 2m'},$$

where z is a root of the equation (which has necessarily a real root)

$$z^3 - (3p^2 - 8q)\,z^2 + (3p^4 - 16p^2q + 16q^2 + 16pr - 64s)\,z - (8r - 4pq + p^3)^2 = 0.$$

Ex. 2. To resolve $x^4 - 6x^2 + nx - 3 = 0$ into its factors. Dividing by $x^2 + ax + b$, we find a remainder

$$(n + 2ab + 6a - a^3)\,x - (a^2b - b^2 - 6b + 3);$$

therefore to determine a and b we have

$$n + 2ab + 6a - a^3 = 0$$

$$a^2b - b^2 - 6b + 3 = 0.$$

Solving the former with respect to b, and substituting in the latter, we have $(a^2 - 4)^3 = n^2 - 64$, or $a = \sqrt{4 + \sqrt[3]{n^2 - 64}}$; from whence b, and the other quadratic factor

$$x^2 - ax + a^2 - b - 6$$

may be determined.

SECTION IX.

ON ELIMINATION.

165. An equation between two unknown quantities x and y, supposed to contain no term which is fractional or irrational, is said to be of the degree which is expressed by the sum of the indices of x and y in that term where the sum is the greatest. The general equation of the n^{th} degree between x and y ought to contain all the terms in which the sum of the indices does not exceed n; therefore, when complete and arranged according to descending powers of x, it will be

$$a_0x^n + (b_0 + b_1y)\,x^{n-1} + (c_0 + c_1y + c_2y^2)\,x^{n-2} + \ldots\ldots$$
$$+ (l_0 + l_1y + l_2y^2 + \ldots\ldots + l_ny^n) = 0.$$

When an equation is incomplete, that is, when it does not contain all the terms which belong to its degree, we must suppose, in the general equation, the coefficients of the deficient terms to be equal to zero.

Although we are always at liberty to divide an equation by any one of its coefficients, we cannot in the above general equation suppose $a_0 = 1$, for then it would not comprehend those equations which want the term involving x^n. After having divided the equation by any one of its coefficients, there will remain as many indeterminate constants as there are

terms, wanting one; the number of these constants will therefore be

$$2 + 3 + 4 + \ldots + (n + 1) = \tfrac{1}{2}n(n + 3),$$

which expresses how many conditions an equation of the n^{th} degree may be made to satisfy, by a suitable determination of its coefficients.

166. To eliminate between two equations of any degree involving two unknown quantities, is to obtain an equation containing only one of the unknown quantities, and which gives all the values of this unknown quantity, which, together with the corresponding values of the other unknown, can satisfy the proposed equations. This equation involving only one unknown quantity is called the final equation, and its roots are called suitable values. In what follows, we shall suppose the polynomials which form the first members of the equations to be freed from any common divisor which they may admit; for if they had a common divisor containing both variables, it might be reduced to zero, and therefore the proposed equations might be satisfied, by an infinite number of systems of values of x and y; or if they had a common divisor containing only one of the variables, there would be a limited number of values of that variable, and an unlimited number of values of the other, by which the proposed equations might be satisfied; so that in both cases there could be no final equation.

167. To determine all the systems of values which will satisfy two equations between two unknown quantities, each being of any degree.

Let $F(x, y) = 0$, $f(x, y) = 0$ be two equations respectively of the m^{th} and n^{th} degrees, admitting only a limited number of pairs of values of x and y, and their first members consequently having no common divisor, involving either both or only one of the variables. Then in order that any value

$y = \beta$ may be a suitable value, it is necessary that there should exist one or more values of x which, substituted in the polynomials $F(x, \beta), f(x, \beta)$, will reduce them to zero; these polynomials must therefore have a common measure a function of x, which, equated to zero, will give one or more values of x, that jointly with $y = \beta$ satisfy the proposed equations. If therefore we perform the operation for finding the greatest common measure of $F(x, y), f(x, y)$, (which we suppose arranged according to descending powers of x) introducing or suppressing factors, functions of y, so that no quotient shall have any term with y in its denominator, we shall at last arrive at a remainder independent of x, which put equal to zero will give the final equation $\psi(y) = 0$. For if β be a root of this equation, and $\phi(x, y)$ be the last divisor, since $y = \beta$ makes the remainder vanish, $\phi(x, \beta)$ is a common measure of the polynomials $F(x, \beta), f(x, \beta)$; therefore $\phi(x, \beta) = 0$ will give values of x which, jointly with $y = \beta$, satisfy the proposed equations.

In those cases where the process for the common measure requires neither the introduction nor suppression of factors, we are certain that the last remainder put equal to zero, or $\psi(y) = 0$, will furnish all the suitable values of y, and no more; but we cannot affirm this in other cases, unless we are certain that the last remainder is unaffected by the factors that have been rejected or introduced; and it frequently happens that in the final equation, values of y are found which are foreign to the problem, and others are deficient which belong to it. On this account the method of elimination by the greatest common measure is imperfect, but it is still the most convenient practical one for numerical equations.

168. In particular cases we are able to find all the systems

of values which satisfy two proposed equations, by easier methods than the one just described. Thus, whenever we are able to solve one of the equations with respect to one of the unknown quantities, x for instance, we have only to substitute the resulting expressions for x in the other equation, and we shall obtain equations containing y only; and if we substitute the values of y given by these equations in the corresponding expressions for x, we shall obtain all the pairs of values required. Also if the two equations are of the same degree with respect to the variable which we wish to eliminate, we may, by introducing factors, if necessary, and subtracting, depress one of them to an inferior degree. And if the first members of the equations are or can be resolved into their factors, then the solution of them is reduced to the simpler case of finding all values of x and y which reduce at the same time a factor of each to zero.

169. In all cases of elimination between the equations

$$\mathrm{F}(x, y) = 0, f(x, y) = 0,$$

besides expelling any common factor which the polynomials admit, the application of the general method may be simplified by previously ascertaining whether either has factors containing only one of the variables. This may be done by arranging each, first according to powers of y, and finding the greatest common measure of the coefficients of the several powers of y in it; and secondly by arranging each according to powers of x, and finding the greatest common measure of the coefficients of the several powers of x. Let X, Y be the factors thus discovered of $\mathrm{F}(x, y)$, and M its remaining factor; and let X′, Y′, N be similar quantities for $f(x, y)$; then the proposed system may be replaced by

$$\mathrm{XYM} = 0, \mathrm{X'Y'N} = 0,$$

which will be satisfied by simultaneously putting any factor of

each equal to zero, provided we do not take X, X′, together, or Y,Y′, together; for X, X′, cannot be reduced to zero by the same value of x, unless they have a common factor; and that they cannot have, since by the supposition $F(x, y)$, $f(x, y)$ have no common factor. Hence, with the exception of $M = 0$, $N = 0$, each system into which $F(x, y) = 0$, $f(x, y) = 0$, is resolved, has at least one of its equations involving only one unknown quantity; and therefore its solution is attended with no other difficulties than what belong to equations of that description. But the system $M = 0$, $N = 0$, whose first members contain both variables, but have no factors depending on x only or y only, will require the process of elimination by the greatest common measure to be applied to them, in order to reduce their solution to that of equations containing only one unknown quantity, as we shall now more minutely explain.

170. To examine the consequences of introducing or suppressing factors in the process of elimination by the greatest common measure, and to investigate the means of obtaining an exact final equation.

Let $M = 0$, $N = 0$, be two equations between x and y of the m^{th} and n^{th} degrees respectively, the polynomials M and N being arranged according to descending powers of x, and not admitting a common divisor, and neither of them having a factor composed of x only, or of y only; and let m be greater than n. Divide M by N, and let Q be the quotient (containing no term with y in its denominator) and R the remainder, so that

$$M = QN + R;$$

then all values of x and y which satisfy $M = 0$, $N = 0$, also satisfy $N = 0$, $R = 0$; but if the division cannot be performed without putting powers of y in the denominator of the quotient, *i.e.* if Q be of the form $\frac{H}{K}$, where K contains y, then

we cannot affirm that all values which satisfy the proposed system also satisfy $N = 0$, $R = 0$; for the equation

$$M = \frac{H}{K} N + R$$

shews that values which make $M = 0$, $N = 0$, may make $K = 0$, so that $\frac{H}{K} N$ may assume the form $\frac{0}{0}$, the real value of which, and therefore of R, may be finite or infinite instead of zero; and, conversely, values which make $N = 0$, $R = 0$, may still not make the second member equal to zero, and therefore not make M equal to zero. To avoid fractional quotients, we must use the same means as in finding the greatest common measure; that is, we must multiply M by the coefficient of the first term of N or by certain factors of that coefficient; then no common factor will have been introduced into both polynomials; and if P represent this multiplier, a function of y, Q the quotient, and R the remainder, we shall have

$$PM = QN + R,$$

which shews that the solutions of $N = 0$, $R = 0$, are the same as those of $PM = 0$, $N = 0$. But these latter equations resolve themselves into the two systems $M = 0$, $N = 0$; $P = 0$, $N = 0$. Therefore, besides the solutions of the proposed equations, the system $N = 0$, $R = 0$, will furnish those of $P = 0$, $N = 0$. Hence we must solve the two latter equations, one of which $P = 0$ contains only y, and substitute all the resulting pairs of values of x and y in $M = 0$; then those pairs of values which do not satisfy it must be rejected, and we shall thus obtain those solutions of $N = 0$, $R = 0$, which belong to the proposed system $M = 0$, $N = 0$. The remaining solutions of the proposed system are contained in the equations $N = 0$, $R = 0$, R being a polynomial of smaller dimensions than N. Now if R have factors containing only

one of the variables, (which may be discovered by seeking the greatest common measure of its coefficients when arranged according to the powers of each variable in succession,) so that $R = XYR'$, then the system $N = 0$, $R = 0$, may be resolved into the three systems

$$N = 0, X = 0; \; N = 0, Y = 0; \; N = 0, R' = 0;$$

the two former of which present no difficulty, because one equation in each contains only one variable; and the third $N = 0$, $R' = 0$, is exactly of the same nature as the one we started with; for N, R', have no common factor, otherwise M and N would have the same common factor, which is contrary to the supposition, and neither N nor R' admits a factor containing only one of the variables. This system then, by exactly the same process, may be replaced by another similar system $R' = 0$, $R'' = 0$, the latter being of a lower degree in x than the former; and the system $R' = 0$, $R'' = 0$, by another, of which the second equation will be of a degree in x, inferior to that of $R'' = 0$. In continuing these uniform operations, we shall at last arrive at a remainder not containing x; suppose this to be $R'' = 0$, then the solution of the proposed system is reduced to that of $R' = 0$, $R'' = 0$, and is thus made to depend upon the solution of an equation containing only one unknown quantity.

171. In ascending from $R' = 0$, $R'' = 0$, to the preceding system $N = 0$, $R' = 0$, it may happen that some solutions will have to be added, and others suppressed; and, similarly, in ascending from $N = 0$, $R' = 0$, to $M = 0$, $N = 0$; and so on, if there was a greater number of successive divisions. This method then, as we perceive, will not always lead to a single equation in y, but to several, some of which may give unsuitable values for that variable. When we have recognized all those which really enter into the solutions common to the two proposed equations, we may, if necessary, combine them

into one final equation. It may be observed that, since M and N are prepared so as to admit no common measure, we can never find zero, but we may find a number, for the last remainder R''; in that case the final equation $R'' = 0$ is absurd; and the proposed equations (unless solutions have been suppressed in the process) are incompatible with one another; *i.e.* incapable of being satisfied by finite values of x and y. It is easy to form equations of this sort; such for instance are $P = 0$, $PQ + k = 0$; P and Q being integral functions of x and y, and k a number; for the condition expressed by the former, reduces the latter to $k = 0$, which is absurd, since k is a number. Also from the final equation $R'' = 0$, we can never deduce a value β, of y, which will reduce the preceding divisor R' to zero independently of the value of x; for in that case R' would have a factor, $y - \beta$, which is impossible, because in the process each remainder, before being employed as a divisor, is cleared of factors containing x only, or y only; but $y = \beta$ may destroy some of the terms in R', and so cause $R' = 0$ to furnish a greater or smaller number of corresponding values of x, or none at all if $y = \beta$ reduce R' to a number. Of these peculiarities, and of the application of the general method, the following are instances :

Ex. 1.
$$yx^3 - (y^3 - 3y - 1)\,x + y = 0$$
$$x^2 - y^2 + 3 = 0.$$

The first division gives the remainder $x + y$; and the division of $x^2 - y^2 + 3$ by $x + y$ gives the remainder 3. The proposed equations are therefore incompatible.

Ex. 2.
$$(y-1)\,x^3 + (y^2+y)\,x^2 + (3y^2+y-2)\,x + 2y = 0$$
$$(y-1)\,x^2 + (y^2+y)\,x + 3y^2 - 1 = 0.$$

The final equations are
$$y^2 - 1 = 0,\ (y-1)\,x + 2y = 0;$$
the former gives $y = \pm 1$; but the value $y = 1$, furnishes no

corresponding finite value of x, since it reduces the latter to $2 = 0$.

Ex. 3.
$$x^3 - 3yx^2 + (3y^2 - y + 1)\,x - y^3 + y^2 - 2y = 0$$
$$x^2 - 2yx + y^2 - y = 0.$$

$$x^2-2yx+y^2-y\big)\,x^3-3yx^2+(3y^2-y+1)x-y^3+y^2-2y\,\big(x-y$$
$$x^3-2yx^2+(y^2-y)x$$
$$\overline{-yx^2+(2y^2+1)x-y^3+y^2-2y}$$
$$-yx^2+2y^2x-y^3+y^2$$
$$\overline{x-2y)\,x^2-2yx+y^2-y\,(x}$$
$$x^2-2yx$$
$$\overline{y^2-y,}$$

therefore the final equations are

$$x - 2y = 0, \quad y^2 - y = 0;$$

$$\text{which give } \left.\begin{aligned} y &= 0 \\ x &= 0 \end{aligned}\right\} \left.\begin{aligned} y &= 1 \\ x &= 2 \end{aligned}\right\};$$

and as no factor has been introduced or suppressed, these two solutions are those of the proposed system.

Ex. 4.
$$(y - 2)\,x^2 - 2x + 5y - 2 = 0$$
$$yx^2 - 5x + 4y = 0.$$

Multiply the dividend by y,

$$yx^2 - 5x + 4y\big)\,(y-2)\,yx^2 - 2yx + 5y^2 - 2y\,\big(y-2$$
$$(y-2)\,yx^2 - (5y - 10)\,x + 4y^2 - 8y$$
$$\overline{(3y - 10)\,x + y^2 + 6y.}$$

Multiply the dividend by $(3y - 10)^2$,

$$
\begin{array}{l}
(3y-10)x+y^2+6y\Big) \\
(3y-10)^2yx^2-5(3y-10)^2x+4(3y-10)^2y\Big((3y-10)yx+\ldots \\
\underline{(3y-10)^2yx^2+(3y-10)(y^2+6y)yx} \\
-(3y-10)(y^3+6y^2+15y-50)x+4(3y-10)^2y \\
\underline{-(3y-10)(y^3+6y^2+15y-50)x-(y^2+6y)(y^3+6y^2+15y-50)} \\
\qquad y^5+12y^4+87y^3-200y^2+100y.
\end{array}
$$

Therefore the final equations are (suppressing the factor y, for the solution $y = 0$, $x = 0$ does not satisfy the proposed system, and is due to the factor introduced in the operation)

$$(3y - 10)\,x + y^2 + 6y = 0,$$

$$y^4 + 12y^3 + 87y^2 - 200y + 100 = 0,$$

which contain no false values; for the only false value which the final equation in y could contain, would be $\frac{10}{3}$, which is impossible, since all the coefficients of that equation are integers. One pair of values is $y = 1$, $x = 1$; the other solutions can be obtained only approximately.

Ex. 5. $$x(4y^2 + 3) - 8ay = 0,$$
$$4y(3 - 2x^2) - 4y^2 + 3 = 0.$$

Here we can solve with respect to one of the variables, and we find for the final equations

$$(x^2 - 1)^3 = a^2 - 1, \quad y = \frac{a}{x} + \frac{3}{2} - x^2;$$

$$\therefore x = \sqrt{1 + \sqrt[3]{a^2 - 1}}, \quad y = \frac{a}{\sqrt{1 + \sqrt[3]{a^2 - 1}}} + \tfrac{1}{2} - \sqrt[3]{a^2 - 1}.$$

172. It was observed (Art. 23) that the problem of transforming an equation, in its widest sense, required the general methods of elimination. This is especially the case where each root of the new equation is to be composed of several roots of the primitive equation. Of this use of the methods of elimination we shall now give some instances.

To transform an equation into one whose roots shall be the differences of every two roots of the proposed equation.

Let $f(x) = 0$ be an equation of n dimensions, having roots $a, b, c, \ldots \ldots l$; to obtain another equation whose roots are the differences between all the roots of the proposed and a, we must make $y = x - a$ or $x = a + y$, and the substitution of this value for x in $f(x) = 0$, will give $f(a + y) = 0$, the required equation; or, developing, (Art. 27),

$$f(a) + f'(a) \cdot y + f''(a) \frac{y^2}{1.2} + \ldots \ldots + y^n = 0. \ldots (1).$$

Since, by the supposition, a is a root of the proposed, $f(a) = 0$; therefore the preceding equation has y for a factor, or admits a root zero, corresponding to the difference $a - a$; suppressing this factor, we have

$$f'(a) + f''(a) \frac{y}{1.2} + \ldots \ldots + y^{n-1} = 0 \ldots \ldots (2),$$

an equation having for its roots the difference between a and the $n - 1$ other roots of the proposed equation. If in this equation we replace a by $b, c, \ldots \ldots$, successively, we shall form equations whose roots are the differences between b and the $n - 1$ other roots, between c and the $n - 1$ other roots, and so on. Hence it follows that the differences of every two of the roots of the proposed equation are the values of y furnished by the equation

$$f'(x) + f''(x) \frac{y}{1.2} + \ldots \ldots + y^{n-1} = 0,$$

when we substitute successively in it, for x, all the roots of the equation $f(x) = 0$; which amounts to solving the system formed by the above equation, and $f(x) = 0$. If therefore we eliminate x between the equations

$$f(x) = 0,\ f'(x) + f''(x)\cdot\frac{y}{1.2} + \ldots\ldots + y^{n-1} = 0,$$

the resulting equation in y will be the one required. The proposed equation being of the n^{th} degree, the transformed equation will be of the $\frac{n(n-1)}{2}^{th}$ degree; for the number of its roots is equal to the number of permutations which can be found with the n quantities $a, b, c, \ldots l$, taken two and two together; also the transformed equation will contain only even powers of y, for if it have a root $a - b$, it will also have the root $b - a$; so that its roots are equal two and two, and of contrary signs. Hence if $n(n-1) = 2m$, and $y^2 = z$, the transformed equation will be of the form

$$z^m + q_1 z^{m-1} + q_2 z^{m-2} + \ldots\ldots + q_m = 0,$$

and the values of z are the squares of the differences of every two roots of the proposed equation.

Ex. $\qquad x^3 + qx + r = 0.$

In this case $f'(x) + f''(x)\cdot\frac{y}{1.2} + y^2 = 0$ gives

$$3x^2 + q + 3xy + y^2 = 0.$$

$$3x^2 + 3xy + y^2 + q)\ 3x^3 + 3qx + 3r\ (x - y$$
$$3x^3 + qx + \ldots.$$
$$\overline{2(y^2 + q)\,x + y^3 + qy + 3r}$$

$$2(y^2+q)x+y^3+qy+3r\Big)$$

$$6\,(y^2+q)^2\,x^2 + 6(y^2+q)^2\,yx + 2(y^2+q)^3\Big(3x(y^2+q) + \ldots$$
$$6(y^2+q)^2 x^2 + 3\,(y^2+q)^2\,yx + \ldots.$$
$$\overline{4(y^2+q)^3 - 3(y^3+qy+3r)\,(y^3+qy-3r)}$$

therefore, equating the last remainder to zero (since the factor $y^2 + q$ put equal to zero, reduces the last divisor to $3r$, which is different from zero), we have the equation of differences

$$y^6 + 6qy^4 + 9q^2y^2 + 4q^3 + 27r^2 = 0;$$

and putting $y^2 = z$, the equation of the squares of the differences is (as at p. 40)

$$z^3 + 6qz^2 + 9q^2z + 4q^3 + 27r^2 = 0.$$

By similar reasoning it may be shewn, that to transform $f(x) = 0$, into one whose roots shall be the sum, product, or ratio of every two of its roots, we must eliminate x between $f(x) = 0$, and

$$f'(x) + f''(x)\frac{h}{1.2} + \dots + h^{n-1} = 0,$$

where $h = y - 2x$, $\dfrac{y}{x} - x$, and $xy - x$ respectively; taking in the two former cases the square root of the result.

173. To eliminate one of the unknown quantities between two equations containing two unknown quantities, by means of symmetrical functions.

$$\text{Let } x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0 \dots (1)$$
$$x^m + q_1x^{m-1} + q_2x^{m-2} + \dots + q_m = 0 \dots (2)$$

be two equations respectively of n and m dimensions in x and y; so that $p_1, p_2, \dots p_n$ are functions of y involving respectively no power of y above the first, second, &c., n^{th}; and $q_1, q_2, \dots q_m$ functions of y involving no power of y above the first, second, &c., m^{th}. If we can solve the first with respect to x, and deduce n values $a, b, c, \dots$, functions of y, then upon substituting them in the second, we shall have to determine y, n equations not containing x, viz.

$$\left.\begin{array}{l} a^m + q_1 a^{m-1} + q_2 a^{m-2} + \ldots\ldots + q_m = 0 \\ b^m + q_1 b^{m-1} + q_2 b^{m-2} + \ldots\ldots + q_m = 0 \\ c^m + q_1 c^{m-1} + q_2 c^{m-2} + \ldots\ldots + q_m = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots = . \end{array}\right\} \ldots (3).$$

But in general the solution of (1) is impossible, and our object must be to obtain a final equation containing indifferently all the suitable values of y, and this we shall do by multiplying together the above n equations; for the result will be satisfied by every value of y derived from any one of them, and by no other quantity; and to every one of these values of y there will correspond a value of x such that the pair will jointly satisfy (1) and (2). For suppose that a value of y deduced from the first of equations (3) is β, and let the equation $x - a = 0$ give, by making $y = \beta$, $x = a$; then it is manifest that $x = a$, $y = \beta$ will jointly satisfy the proposed equations. But in the result of this multiplication, the factors only change places when we interchange in any manner the quantities $a, b, c, \ldots\ldots$; therefore the product will only involve rational and integral symmetrical functions of these quantities, which may be expressed by means of the coefficients of equation (1); and we shall so obtain the final equation in y. The calculations required by this method are in general tedious; but it has the recommendation of giving the final equation with all the roots it ought to contain, and no more.

174. When we eliminate one of the unknown quantities between two equations containing two unknown quantities, the degree of the final equation cannot exceed the product of the degrees of the two equations between which the elimination is performed.

To prove this, we must examine to what degree y may rise in the symmetrical functions composing the product of equations (3). Each term of this product will be itself the product of

terms, one taken out of each of the equations (3), and will therefore be of the form

$q_{m-h}a^h \times q_{m-k}b^k \times q_{m-l}c^l \ldots$, or $q_{m-h} \times q_{m-k} \times q_{m-l} \ldots \times a^h b^k c^l \ldots$

But the product of the n equations, being symmetrical, must contain all the terms of the same form which we can make with the above quantities; consequently it will contain all the terms represented by

$$q_{m-h}\, q_{m-k}\, q_{m-l} \ldots \Sigma(a^h b^k c^l \ldots) \quad (4),$$

and we must now ascertain the dimensions of this expression.

Now the degree of y in $q_{m-h}, q_{m-k}, \ldots$ cannot exceed $m-h, m-k, \ldots$ respectively; therefore in $q_{m-h}\, q_{m-k}\, q_{m-l} \ldots$ it will at most be equal to $mn-h-k-l- \ldots$. Also if we refer to the formulæ which give the values of the double, triple, &c. functions in terms of the sums of the powers of the roots, we see that in $\Sigma(a^h b^k c^l \ldots)$ the term of highest dimension in y will be found in $S_h S_k S_l \ldots$; but the equations which give $S_1, S_2, \ldots$ in terms of $p_1, p_2, \ldots$ (since these quantities do not involve powers of y exceeding the first, second, third, &c. respectively) shew that the degree of y in any sum S_h cannot exceed h; therefore the degree of y in $\Sigma(a^h b^k c^l \ldots)$ cannot surpass $h+k+l+\ldots$; consequently in the expression (4). the degree of y will at the most be equal to mn. The same thing may be similarly proved of all the symmetrical functions whose sum makes up the product of the n equations. Therefore, lastly, the degree of the final equation cannot exceed the product of the degrees of the two equations from which it results by the elimination of one of the unknown quantities.

Although the degree of the final equation cannot exceed mn, in particular cases it may be less than mn. If we extend the process to any number of equations, we shall have the general theorem discovered by *Bezout*, viz. that if between equations

equal in number to the unknown quantities we eliminate all except one, the degree of the final equation will be at most equal to the product of the degrees of the several equations.

Ex. To eliminate x between the equations

$$yx^2 - 5x + 4y = 0$$
$$(y-2)\,x^2 - 2x + 5y - 2 = 0.$$

Let a and b denote the values of x given by the first equation; then substituting them in the second equation, we have

$$(y-2)\,a^2 - 2a + 5y - 2 = 0$$
$$(y-2)\,b^2 - 2b + 5y - 2 = 0;$$

the product of these equations, which will be the required final equation in y, is

$$(y-2)^2\,\Sigma(a^2b^2) - 2(y-2)\Sigma(a^2b) + (y-2)(5y-2)S_2$$
$$- 2(5y-2)\,S_1 + 4\Sigma(ab) + (5y-2)^2 = 0.$$

But $p_1 = -\dfrac{5}{y}$, $p_2 = 4$, $p_3 = 0$;

$$\therefore\ S_1 = \frac{5}{y},\ S_2 = \frac{25 - 8y^2}{y^2},\ S_3 = \frac{125 - 6y^2}{y^3},$$

$$\Sigma(ab) = 4,\ \Sigma(a^2b) = \frac{20}{y},\ \Sigma(a^2b^2) = 16.$$

Hence, substituting and reducing, we find for the final equation (as at p. 179).

$$y^4 + 12y^3 + 87y^2 - 200y + 100 = 0.$$

175. As all the preceding methods suppose the equations to which they are applied to be rational, it is of importance to be able to reduce an equation involving radicals, to a rational form. The extermination of radicals, considered generally, is only a case of elimination, as will appear from the following example.

To reduce $x - \sqrt{x-1} + \sqrt[3]{x+1} = 0$ to a rational form.

Let $$y^2 = x - 1, \quad z^3 = x + 1\,;$$

$$\therefore\ x - y + z = 0;$$

this gives $y = x + z$, and therefore $y^2 = x - 1$ gives

$$z^2 + 2zx + x^2 - x + 1 = 0,$$

and it remains to eliminate z between this and $z^3 - x - 1 = 0$. Using the process of the greatest common measure, we find for the exact final equation,

$$x^6 - 3x^5 + 8x^4 + x^3 + 7x^2 - 7x + 2 = 0;$$

a result that may also be obtained directly from the proposed equation, by successive involutions.

SECTION X.

ON THE GENERAL SOLUTION OF EQUATIONS.

176. A remarkable application of the theory of symmetrical functions is that made by *Lagrange* to the general solution of equations; by that means he solves the general equations of the first four degrees, by a uniform process, and one which includes all others that have been proposed for that purpose, the common relation of which to one another is thus made apparent. It consists in employing an auxiliary equation, called a reducing equation, whose root is of the form

$$x_1 + ax_2 + a^2x_3 + \ldots. + a^{n-1}x_n,$$

denoting by x_1, x_2, ... x_n the n roots of the proposed equation, and by a one of the n^{th} roots of unity; and the principle on which it is based is as follows. Let y be the unknown quantity in the reducing equation, and let

$$y = a_1x_1 + a_2x_2 + \ldots. + a_nx_n,$$

a_1, a_2, ... a_n denoting certain constant quantities; then if $n - 1$ values of y, and suitable values of the constants a_1, a_2, ... a_n can be found, so that we may have $n - 1$ simple equations; these, together with the equation

$$-p_1 = x_1 + x_2 + \ldots. + x_n,$$

will enable us to determine the n roots. Now since in the

expression $a_1x_1 + a_2x_2 + \ldots . + a_nx_n$, there is nothing to distinguish one root from another, $x_1, x_2, \ldots . x_n$ may be permuted in all possible ways, and therefore the expression will have $n\,(n-1) \ldots . 3.2.1$ values, and the equation for determining y will rise to the same number of dimensions, or will be of a degree higher than that of the proposed equation; hence the method will be of no use, unless such values can be assumed for the constants $a_1, a_2, \ldots . a_n$ as shall make the solution of the equation in y depend upon that of an equation at most of $n - 1$ dimensions. Now this may be done (at least when n does not exceed 4) by taking the n^{th} roots of unity $a^\circ, a, a^2, a^3, \ldots . a^{n-1}$ for $a_1, a_2, \ldots . a_n$, so that

$$y = a^\circ x_1 + ax_2 + \ldots + a^{r-1}x_r + a^r x_{r+1} + \ldots + a^{n-1}x_n.$$

For, in the first place, with this assumption, the reducing equation will contain only powers of y, which are multiples of n; for, since $a^n = 1$,

$$a^{n-r}y = a^{n-r}x_1 + a^{n-r+1}x_2 + \ldots x_{r+1} + ax_{r+2} + \ldots a^{n-r-1}x_n,$$

$$\text{or } a^{n-r}y = a^\circ x_{r+1} + ax_{r+2} + \ldots + a^{n-1}x_r,$$

which is the same result as if we had interchanged x_1 and x_{r+1}, x_2 and $x_{r+2}, \ldots .$ so that if y be a root of the reducing equation, $a^{n-r}y$ is also a root; therefore the reducing equation since it remains unaltered when $a^{n-r}y$ is written for y, contains only powers of y which are multiples of n; if therefore we make $y^n = z$, we shall have a reducing equation in z of only $1.2.3 \ldots (n-1)$ dimensions, whose roots will be the different values of z which result from the permutations of the $n - 1$ roots $x_2, x_3, \ldots . x_n$ among themselves. We shall now have, expanding and reducing,

$$z = y^n = u_\circ + u_1 a + u_2 a^2 + \ldots u_{n-1}a^{n-1},$$

in which $u_\circ, u_1, u_2, \ldots . u_{n-1}$ are determinate functions of the roots, which will be invariable for the simultaneous changes of

x_1 into x_{r+1}, x_2 into x_{r+2}, ... since $z = (a^r y)^n$; and when their values are known in terms of the coefficients of the proposed equation, we shall immediately know the values of the roots. For let z_0, z_1, z_2, ... z_{n-1} be the different values of z when $1, a, \beta, \gamma, \ldots. \lambda$, the roots of $y^n - 1 = 0$, are substituted for a; then since $y = \sqrt[n]{z}$, we have

$$x_1 + x_2 + \ldots + x_n = \sqrt[n]{z_0}$$
$$x_1 + ax_2 + \ldots. + a^{n-1}x_n = \sqrt[n]{z_1}$$
$$\ldots\ldots\ldots\ldots = \ldots$$
$$x_1 + \lambda x_2 + \ldots. + \lambda^{n-1}x_n = \sqrt[n]{z_{n-1}};$$

therefore, adding, and taking account of the properties of the sums of the powers of $1, a, \beta, \gamma, \ldots.$ (Art. 154), we have

$$nx_1 = \sqrt[n]{z_0} + \sqrt[n]{z_1} + \ldots. + \sqrt[n]{z_{n-1}}.$$

Again, multiplying the above system of equations respectively by $1, a^{n-1}, \beta^{n-1}, \ldots. \lambda^{n-1}$, we have

$$nx_2 = \sqrt[n]{z_0} + a^{n-1}\sqrt[n]{z_1} + \beta^{n-1}\sqrt[n]{z_2} + \ldots + \lambda^{n-1}\sqrt[n]{z_{n-1}},$$

and so on for the rest. Hence, since $-p_1 = \sqrt[n]{z_0}$, and $\therefore (-p_1)^n = z_0 = u_0 + u_1 + \ldots. + u_{n-1}$, the problem is reduced to finding the values of $u_1, u_2, \ldots. u_{n-1}$.

177. When n is a composite number, the above general method admits of simplifications. For let n have a divisor m so that $n = mp$, and let a be a root of $y^m - 1 = 0$; then since $a^m = 1$, $a^{m+1} = a$, $a^{m+2} = a^2, \ldots a^{2m} = 1$, $a^{2m+1} = a, \ldots$ we have

$$y = x_1 + ax_2 + a^2x_3 + \ldots + a^{n-1}x_n$$
$$= X_1 + aX_2 + a^2X_3 + \ldots + a^{m-1}X_m,$$

where $X_r = x_r + x_{m+r} + x_{2m+r} + \ldots + x_{n-m+r}$, and consists of p roots;

$$\therefore z = y^m = u_0 + u_1 a + u_2 a^2 + \ldots + u_{m-1}a^{m-1},$$

where u_0, u_1, ... are known functions of X_1, X_2, ...; and when they are found in terms of the coefficients of the proposed equation, we shall be able to determine immediately the values of X_1, X_2, ... as before. To deduce the values of the primitive roots x_1, x_2, x_3 .... x_n, we must regard separately those which compose each of the quantities X_1, X_2, ... as the roots of an equation of p dimensions. Thus let the roots whose sum is X_1, be those of the equation

$$x^p - X_1 x^{p-1} + Lx^{p-2} - Mx^{p-3} + \ldots\ldots = 0$$

where L, M, .. are unknown; then the first member of this equation is a divisor of the first member of the proposed, since all its roots belong to the latter. Hence, effecting the division and equating to zero the coefficients of x^{p-1}, x^{p-2} .. in the remainder, we shall have p equations in X_1, L, M, ... of which the $p-1$ first will give the values of L, M, ... in terms of X_1 by linear equations. It will then remain to solve the equation so formed of p dimensions. Similarly, substituting the value of X_2 in place of that of X_1, we shall have an equation giving the next group of roots x_2, x_{m+2} .... ; and so on.

Ex. 1. $$x^3 - px^2 + qx - r = 0.$$

Let its roots be a, b, c, and let

$$y = a + ab + a^2c;$$

$$\therefore z = y^3 = a^3 + b^3 + c^3 + 6abc + 3(a^2b + b^2c + c^2a)\,a + 3(a^2c + b^2a + c^2b)\,a^2,$$

$$= u_0 + u_1 a + u_2 a^2.$$

But u_1, u_2 are roots of the quadratic

$$u^2 - (u_1 + u_2)\,u + u_1u_2 = 0,$$

and $$u_1 + u_2 = 3\Sigma(a^2b) = 3pq - 9r$$

$$u_1u_2 = 9\,\{abcS_3 + \Sigma(a^3b^3) + 3a^2b^2c^2\}$$

$$= 9q^3 + 9(p^3 - 6pq)\,r + 81r^2.$$

Hence u_1, u_2 are known,

and $\therefore u_0 = p^3 - (u_1 + u_2)$ is known.

Hence, denoting by z_1, z_2, the values of z when α and α^2 are respectively written for α, we have

$$a + b + c = p$$

$$a + \alpha b + \alpha^2 c = \sqrt[3]{z_1}$$

$$a + \alpha^2 b + \alpha c = \sqrt[3]{z_2},$$

from which we obtain the values of a, b, and c, viz.

$$a = \tfrac{1}{3}(p + \sqrt[3]{z_1} + \sqrt[3]{z_2})$$

$$b = \tfrac{1}{3}(p + \alpha^2 \sqrt[3]{z_1} + \alpha \sqrt[3]{z_2})$$

$$c = \tfrac{1}{3}(p + \alpha \sqrt[3]{z_1} + \alpha^2 \sqrt[3]{z_2}).$$

Ex. 2. $\quad x^4 - px^3 + qx^2 - rx + s = 0.$

Since $4 = 2.2$, let α be a root of $y^2 - 1 = 0$, so that $\alpha^2 = 1$,

$$\text{then } y = x_1 + \alpha x_2 + x_3 + \alpha x_4 = \mathrm{X}_1 + \alpha \mathrm{X}_2$$

$$\text{if } \mathrm{X}_1 = x_1 + x_3, \qquad \mathrm{X}_2 = x_2 + x_4;$$

$$\therefore z = y^2 = u_0 + \alpha u_1$$

where $u_0 = \mathrm{X}_1^2 + \mathrm{X}_2^2$, $u_1 = 2\mathrm{X}_1\mathrm{X}_2$, and $u_0 + u_1 = z_0 = p^2$.

Hence $u_1 = 2(x_1 + x_3)(x_2 + x_4)$, by interchanging the roots among themselves, will admit the two other values $2(x_1 + x_2)(x_3 + x_4)$, and $2(x_1 + x_4)(x_2 + x_3)$, and will therefore be a root of an equation of the form

$$u_1^3 - \mathrm{M}u_1^2 + \mathrm{N}u_1 - \mathrm{P} = 0,$$

the coefficients being symmetrical functions of x_1, x_2, x_3, x_4, and consequently determinable in p, q, r, s. It is easily seen that if we make $u_1 = 2q - 2u$, we shall have an equation in u whose roots are

$$x_1x_3 + x_2x_4, \quad x_1x_2 + x_3x_4, \quad x_1x_4 + x_2x_3;$$

and the transformed equation is (Art. 162)

$$u^3 - qu^2 + (pr - 4s)\,u - (p^2 - 4q)\,s - r^2 = 0.$$

Let u' be a root of this equation, then $u_1 = 2q - 2u'$; hence, making

$$a = -1,\ z_1 = u_0 - u_1 = p^2 - 2u_1 = p^2 - 4q + 4u';$$

$$\therefore\ X_1 + X_2 = p,\quad X_1 - X_2 = \sqrt{z_1};$$

$$\therefore\ X_1 = \tfrac{1}{2}(p + \sqrt{z_1}),\quad X_2 = \tfrac{1}{2}(p - \sqrt{z_1}).$$

Hence x_1, x_3, may be regarded as roots of a quadratic $x^2 - X_1x + L = 0$; dividing the proposed by this, and putting the first term of the remainder equal to zero, we find

$$L = \frac{X_1^3 - pX_1^2 + qX_1 - r}{2X_1 - p};$$

therefore x_1, x_3, are known; and x_2, x_4, will result from the same formulæ by interchanging X_1 and X_2, or by changing the sign of the radical $\sqrt{z_1}$.

Ex. 3. $\dfrac{x^n - 1}{x - 1} = 0$, n being a prime number.

If r be one of the roots, and a be a primitive root of the prime number n, (that is, a number whose several powers from 1 to $n - 1$, when divided by n, leave different remainders) it is proved (Art. 80) that all the roots of this equation may be represented by

$$r,\ r^a,\ r^{a^2},\ r^{a^3},\ \ldots\ r^{a^{n-2}}.$$

Let $y = r + ar^a + a^2r^{a^2} + \ldots + a^{n-2}r^{a^{n-2}}$,

a being a root of the equation $y^{n-1} - 1 = 0$. Therefore, observing that $a^{n-1} = 1$ and $r^n = 1$,

$$z = y^{n-1} = u_0 + au_1 + a^2u_2 + \ldots + a^{n-2}u_{n-2}, \ldots (1),$$

u_0, u_1, ... being rational and integral functions of r which do not change by the substitution of r^a, r^{a^2}, r^{a^3}, ... in the place of r; for these quantities regarded as functions of x_1, x_2, x_3, ... do

not alter by the simultaneous changes of x_1 into x_2, x_2 into x_3, . . . nor by the simultaneous changes of x_1 into x_3, x_2 into x_4, . . ., to which correspond the changes of r into r^a, into r^{a^2}, . .

Now every rational and integral function of r, in which $r^n = 1$, may be reduced to the form

$$A + Br + Cr^2 + Dr^3 + \dots + Nr^{n-1},$$

the coefficients A, B, C, ... N being given quantities independent of r; or since in this case the powers $r, r^2, r^3, \dots r^{n-1}$ may be represented although in a different order by r, r^a, $r^{a^2}, \dots r^{a^{n-2}}$, we may reduce every rational function of r to the form

$$A + Br + Cr^a + Dr^{a^2} + \dots + Nr^{a^{n-2}}.$$

Therefore, if this function is such that it remains unaltered when r is changed into r^a, it follows that the new form

$$A + Br^a + Cr^{a^2} + Dr^{a^3} + \dots + Nr$$

coincides with the preceding,

$$\therefore B = C,\ C = D,\ D = E, \dots N = B,$$

and therefore the function is reduced to the form

$$A + B\,(r + r^a + r^{a^2} + \dots r^{a^{n-2}}) \text{ or } A - B,$$

since the sum of the roots $= -1$; hence each of the quantities u_0, u_1, u_2, . . . will be of the form A — B, and its value will be found by the actual development of $z = y^{n-1}$; so that we have the case where the values of u_0, u_1, u_2, . . . are known immediately, without depending upon the solution of any equation. Hence if we denote by $1, a, \beta, \gamma, \dots$ the $n - 1$ roots of the equation $x^{n-1} - 1 = 0$, and by z_0, z_1, z_2, . . . the value of z answering to the substitution of these roots in the place of a in equation (1), we shall have, as in the former cases,

$$r = \frac{\sqrt[n-1]{z_0} + \sqrt[n-1]{z_1} + \sqrt[n-1]{z_2} + \dots + \sqrt[n-1]{z_{n-1}}}{n-1},$$

an expression for one of the roots of the equation $x^n - 1 = 0$; and the other roots are r^2, r^3, &c.

Thus the solution of $x^n - 1 = 0$ is reduced to that of the inferior equation $y^{n-1} - 1 = 0$, of which $1, a, \beta, \gamma, \ldots$ are the roots; also since $n - 1$ is a composite number, the determinator of $a, \beta, \gamma, \ldots.$ will not require the solution of an equation of a higher degree than the greatest prime number in $n - 1$; that is, the solution of $x^n - 1 = 0$ (n prime) may be made to depend upon the solution of equations whose degrees do not exceed the greatest prime number which is a divisor of $n - 1$.

Ex. 4. $\qquad x^5 - 1 = 0.$

The least primitive root of 5 is 2, for its powers from 1 to 4 leave remainders 2, 4, 3, 1;

$$\therefore y = r + ar^2 + a^2r^4 + a^3r^3,$$

$$\text{also } a^4 = 1, \quad r^5 = 1, \quad \text{and } r + r^2 + r^4 + r^3 = -1;$$

$$\therefore z = y^4 = -1 + 4a + 14a^2 - 16a^3.$$

But the four roots of

$$y^4 - 1 = 0 \text{ are } 1, \quad -1, \quad \sqrt{-1}, \quad -\sqrt{-1};$$

$$\therefore z_0 = 1, \quad z_1 = 25, \quad z_2 = -15 + 20\sqrt{-1},$$

$$z_3 = -15 - 20\sqrt{-1};$$

$$\therefore x = \tfrac{1}{4}\{-1 + \sqrt{5} + \sqrt[4]{-15 + 20\sqrt{-1}} + \sqrt[4]{-15 - 20\sqrt{-1}}\}.$$

178. For the proof that, in the general equation of the n^{th} degree, the formation of the reducing equation will require the solution of an equation of $1.2.3 \ldots (n - 2)$ dimensions, when n is prime; and of $\dfrac{1.2.3 \ldots n}{(m-1)\, m\, (1.2.3 \ldots p)^m}$ dimensions, when n is a composite number, and $= mp$, where m is prime; and

that consequently the method fails when n exceeds 4; the reader is referred to Lagrange's *Traité de la resolution des équations numériques*, note xiii., from which the matter of this section is taken.

THE END.

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